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Applying perturbation methods to incomplete market models with exogenous borrowing constraints*

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Abstract

This paper solves an incomplete market model with infinite number of agents and exogenous borrowing constraints described in den Haan, Judd and Juillard (2004). We apply the idea of "barrier methods" to convert optimization problem with borrowing constraints as inequalities into a problem with equality constraints, and the converted model is solved by a second-order perturbation method. The simulation results of impulse responses and second moments match the standardized features of incomplete market models. Accuracy of the solution is in a reasonable range but significantly decreases when the economy is near the borrowing limit or moves away from the steady state.

- JEL Classification: C63, C68, C88, F41;
- Key Words: perturbation, barrier method, borrowing constraint, incomplete market, accuracy.

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1 Introduction

This paper describes how to use perturbation methods to solve incomplete market models with exogenous borrowing constraints, as described in "Problem C" of the JEDC Numerical Methods Comparison project (den Haan, Judd, and Juillard, 2004). In particular, we solve an infinite agent model with incomplete asset markets where agents trade risk-free one-period bonds only. Agents face borrowing constraints that are exogenously given. There are idiosyncratic shocks as well as aggregate shocks in the endowment process, where both shocks have a continuous support.

Using a perturbation method to solve this model can be a challenge because of the following three properties of the model. First, there are infinite number of agents in the model. Applying perturbation methods to an infinite agent model demands a novel implementation of the solution method. Second, the existence of exogenous borrowing constraints makes it hard to use numerical methods associated with equality constraints. Third, the incomplete market model is locally nonstationary by nature.

We overcome these problems as follows. First, in order to solve the infinite-agent problem, we calculate the first order conditions of the representative agent under exogenous asset price. We then derive the process of asset price by using the market clearing condition (i.e., the sum of all asset holdings is equal to zero). We use the linear approximation in the derivation of the analytic solution for asset price, since analytically tractable closed form solution is not available for higher order approximations. Finally, we complete the first order conditions of the representative agent in the infinite agent model by using this linear asset price process.[1] Since the analytic solution is not available for this system, we numerically approximate the model using the second-order perturbation method (quadratic approximation). For algorithms, we use Matlab code `gensys2.m` (Sims, 2001; Kim, Kim, Schaumburg, and Sims, 2003).

In order to deal with hard borrowing constraints, we "soften" the borrowing constraints by modifying utility function so that agents are penalized when borrowing moves close to the "barrier" set by the exogenous bound.[2] With this modified utility function, we can convert an optimization problem with inequality constraints (due to hard borrowing constraints) into an optimization problem with only equality constraints, which allows us to use

[1] This is similar in spirit to the idea of bounded rationality, under which only some moments of variables are considered as in Krusell and Smith (1998).

[2] This method is called "barrier method." We thank Ken Judd for suggesting this modification.

standard numerical solution methods.[3] When applying perturbation method to this model, we use a specific perturbation variable for bond holding so that borrowing limit is never hit in the optimal solution.[4] Utility modification has an additional positive feature that it makes the model stationary, which allows us to derive unconditional moments of variables.

The simulation results suggest the second moments and impulse responses of the model match the standard characteristics of the incomplete market models in international macroeconomics.[5] Accuracy tests based on Euler equation errors suggest that approximation errors increase with tighter borrowing constraint and that quadratic approximation generates smaller errors than linear approximation.

The remaining sections consist of the following. In section 2, we introduce the model and explain how we apply barrier methods to incorporate the borrowing constraints. Section 3 discusses the linearized and quadratic solutions. In section 4, we evaluate the performance of the perturbation solution in three categories. First, we derive impulse responses when aggregate and idiosyncratic shocks hit the economy. Then, we check the properties of the model by deriving second moments of key variables including wealth distribution. Finally, for accuracy test, we compare Euler equation errors from linear and quadratic solutions under difference parameter values. Section 5 concludes the paper.

2 Model

We first present the original model with inequality constraints and then modify the utility function of the model to replace the inequality constraints with equality constraints.

[3] Several papers have studied models with borrowing constraints. Some examples with exogenous bound are Huggett (1993), Levine and Zame (2002), and Kubler and Schmedders (2001). Alternatively, others solve the models with endogenously derived bounds on asset holdings, e.g. endogenous solvency constraints as in Alvarez and Jermann (2000) and enforcement constraints as in Kehoe and Perri (2002).

[4] This is analogous to the way that nonnegativity constraint of consumption is incorporated in macro models by approximating the model with respect to log consumption, which guarantees that optimal solution for consumption never takes a negative value.

[5] For example, Mendoza (1991) and Baxter and Crucini (1995).

2.1 The Original Model

Each agent i maximizes

$$\max \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} \tag{1}$$

subject to

$$c_t^i + q_t b_t^i = y_t^i a_t + b_{t-1}^i. \tag{2}$$
$$b_t^i \geq -\bar{b} \tag{3}$$

where y_t^i is the idiosyncratic shock, a_t is the aggregate shock that is same for each agent, and $\bar{b}$ is the lower bound of bond holdings in borrowing constraints.

The first-order conditions for each agent i can be described as the two-part Kuhn-Tucker conditions;

$$q_t \left(c_t^i\right)^{-\gamma} \geq \beta \mathrm{E_t} \left(c_{t+1}^i\right)^{-\gamma}, \tag{4}$$
$$\left(b_{t+1}^i + \bar{b}\right)\left(q_t \left(c_t^i\right)^{-\gamma} - \beta \mathrm{E_t} \left(c_{t+1}^i\right)^{-\gamma}\right) = 0, \tag{5}$$

Equilibrium requires the world resource constraint that the sum of bond holdings over all agents is equal to zero.

$$\sum_{i=1}^{\infty} b_t^i = 0. \tag{6}$$

The driving processes for idiosyncratic and aggregate shocks are

$$\log(y_t^i) = \frac{-0.5(1-\rho_y)\sigma_y^2}{(1-\rho_y^2)} + \rho_y \log(y_{t-1}^i) + \sigma_y e_{y,t}, \tag{7}$$
$$\log(a_t) = \frac{-0.5(1-\rho_a)\sigma_a^2}{(1-\rho_a^2)} + \rho_a \log(a_{t-1}) + \sigma_a e_{a,t}, \tag{8}$$

where $e_{y,t}$ and $e_{a,t}$ are i.i.d. random variables with a standard Normal distribution.

2.2 A Modified Model

We incorporate borrowing constraint into optimization problem with equality constraints by modifying utility function as follows: Agents are penalized when they borrow or lend from others. The key assumption is that the penalty diverges to infinity as their borrowing approaches the borrowing limit. In particular, we use the following utility function,

$$U(c_t^i, b_t^i) = \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} + \zeta\bar{b}\left[\bar{b}\log\left(b_t^i + \bar{b}\right) - b_t^i\right] \tag{9}$$

Due to the term $\left[\log\left(b_t^i + \bar{b}\right)\right]$ in the utility function, disutility from borrowing significantly increases when agents borrow too much and b_t^i goes towards $-\bar{b}$. Therefore, optimal solution for the borrowing would not exceed this limit. By adjusting the value of ζ, we can adjust how much weight the agents impose on borrowing constraints.[6]

Each agent maximizes the discounted sum of this modified utility function subject to the original budget constraint (2). That is, we convert an optimization problem with inequality constraints into an optimization problem with equality constraints. Euler equation for each agent i becomes

$$q_t\left(c_t^i\right)^{-\gamma} = \beta \mathrm{E_t}\left(c_{t+1}^i\right)^{-\gamma} - \zeta\left(\frac{b_t^i\bar{b}}{b_t^i + \bar{b}}\right) \tag{10}$$

3 Solution

In order to solve the infinite agent model, we first solve the representative agent model given exogenous asset price q_t. Then, we derive the solution for asset price by using the market clearing condition (6). Note that we use linear approximation to derive asset price. Using this asset price process, we can complete the first order conditions for the representative agent.

We derive approximate solutions of the modified model by using a perturbation method. To guarantee the variables stay within the sensible range,

[6] The solution of this modified model converges to that of the original model as $\zeta \to 0$. This is an application of barrier methods, which are a practical way of solving constrained optimization problems. Penalty methods—which attributes penalty outside the domain of the problem—are also widely used. See Judd (1998) and Luenberger (1973) for more on these two methods. Our barrier function is chosen such that its first derivative implies the steady state being independent of ζ.

we use the following variables for perturbation:

$$\begin{aligned}\hat{c}_t^i &= \log c_t^i,\\ \hat{a}_t &= \log a_t,\\ \hat{y}_t^i &= \log y_t^i,\\ \hat{q}_t &= \log\left(\frac{q_t}{\beta}\right),\\ \hat{b}_t^i &= \bar{b}\log\left(\frac{b_t^i+\bar{b}}{\bar{b}}\right).\end{aligned}$$

The last transformation implies that b_t^i cannot move below the exogenous bound.[7]

3.1 Linear Asset Pricing Rule

In this section, we derive the closed-form first-order conditions of the model using linear approximation. We solve the linearized model using eigenvalue decomposition. The linearized version of the first order conditions (budget constraint and Euler equation) for each agent i can be expressed in the following linear system (assuming q_t is exogenously given),

$$\begin{bmatrix} 1 & \beta \\ \gamma & 0 \end{bmatrix}\begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \gamma & -\zeta \end{bmatrix}\begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} \hat{y}_t^i+\hat{a}_t \\ -\hat{q}_{t-1} \end{bmatrix}. \tag{11}$$

It is straightforward to show, as in the Appendix, that the solution for agent i is

$$\frac{\hat{b}_t^i}{\lambda} = \hat{b}_{t-1}^i+\hat{y}_t^i+\hat{a}_t-\sum_{s=0}^{\infty}\beta^s \mathrm{E}_t\left[(1-\beta)\left(\hat{y}_{t+s+1}^i+\hat{a}_{t+s+1}\right)+\frac{\hat{q}_{t+s}}{\gamma}\right] \tag{12}$$

$$\frac{\hat{c}_t^i}{\lambda} = \left(1-\lambda\beta+\frac{\zeta}{\gamma}\right)\left(\hat{b}_{t-1}^i+\hat{y}_t^i+\hat{a}_t\right) + \sum_{s=0}^{\infty}\beta^{s+1}\mathrm{E}_t\left[(1-\beta)\left(\hat{y}_{t+s+1}^i+\hat{a}_{t+s+1}\right)+\frac{\hat{q}_{t+s}}{\gamma}\right], \tag{13}$$

where

$$\lambda=\frac{1}{2}\left[\left(1+\beta^{-1}+\beta^{-1}\gamma^{-1}\zeta\right)-\sqrt{\left(1+\beta^{-1}+\beta^{-1}\gamma^{-1}\zeta\right)^2-4\beta^{-1}}\right].$$

[7] This transformation satisfies three satisfactory properties of the transformation $\hat{b}_t^i = f\left(b_t^i\right)$. They are $f(0)=0$, $f'(0)=1$, and $f''(0)=-\frac{1}{\bar{b}}$.

The next step is to derive the solution for asset price using the linearized first order condition (12) and the market clearing condition (6).[8] Then, we have:

$$0 = 0 + \hat{a}_t - \sum_{s=0}^{\infty} \beta^s \mathrm{E}_t \left[(1-\beta)\, \hat{a}_{t+s+1} + \frac{\hat{q}_{t+s}}{\gamma} \right] \tag{14}$$

whose solution is

$$\hat{q}_t = \gamma \left[\hat{a}_t - \mathrm{E}_t \left(\hat{a}_{t+1} \right) \right] = \gamma \left(1 - \rho_a \right) \hat{a}_t. \tag{15}$$

We use this linear asset price process to complete the representative agent solution in the infinite agent model. Plugging (15) into (12) and (13), we have the following equilibrium solutions:

$$\frac{\hat{b}_t^i}{\lambda} = \hat{b}_{t-1}^i + \hat{y}_t^i - (1-\beta) \sum_{s=0}^{\infty} \beta^s \mathrm{E}_t \left(\hat{y}_{t+s+1}^i \right), \tag{16}$$

$$\begin{aligned} \frac{\hat{c}_t^i}{\lambda} &= \left(1 - \lambda\beta + \frac{\zeta}{\gamma} \right) \left(\hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t \right) \\ &\quad + (1-\beta) \sum_{s=0}^{\infty} \beta^{s+1} \mathrm{E}_t \left(\hat{y}_{t+s+1}^i \right) + \beta \hat{a}_t, \end{aligned} \tag{17}$$

Using the AR(1) shock processes in (7) and (8), we can derive the linearized closed-form solutions for $\hat{b}_t^i$ and $\hat{c}_t^i$:

$$\frac{\hat{b}_t^i}{\lambda} = \hat{b}_{t-1}^i + \left(\frac{1-\rho_y}{1-\beta\rho_y} \right) \hat{y}_t^i, \tag{18}$$

$$\frac{\hat{c}_t^i}{\lambda} = \left(1 - \lambda\beta + \frac{\zeta}{\gamma} \right) \hat{b}_{t-1}^i + \left(\frac{1-\beta}{1-\beta\rho_y} \right) \hat{y}_t^i + \hat{a}_t. \tag{19}$$

3.2 Quadratic Solution

Using the linear asset pricing rule derived in the previous section, we can complete the equation system for the representative agent in the infinite

[8] Second order approximation of $\hat{q}_t$ involves double summation of quadratic and cross product terms and therefore is not analytically tractable.

agent model using quadratic approximation;

$$
\begin{aligned}
0 &= \hat{b}_{t-1}^{i} + \hat{y}_{t}^{i} + \hat{a}_{t}^{i} - \frac{\left(\hat{c}_{t}^{i}\right)^{2}}{2} - \beta \hat{q}_{t} \hat{b}_{t}^{i} + \\
&\quad \frac{\left(\hat{y}_{t}^{i}\right)^{2}}{2} + \frac{\left(\hat{a}_{t}^{i}\right)^{2}}{2} - \hat{c}_{t}^{i} - \beta \hat{b}_{t}^{i}, \qquad (20) \\
0 &= \hat{q}_{t} + \frac{\left(\hat{q}_{t}\right)^{2}}{2} - \gamma \hat{c}_{t}^{i} + \frac{\gamma^{2}\left(\hat{c}_{t}^{i}\right)^{2}}{2} - \gamma \hat{c}_{t}^{i} \hat{q}_{t} \\
&\quad - \mathrm{E}_{t}\left[-\gamma \hat{c}_{t+1}^{i} + \frac{\gamma^{2}\left(\hat{c}_{t+1}^{i}\right)^{2}}{2}\right] + \zeta \hat{b}_{t}^{i} - \frac{\left(\hat{b}_{t}^{i}\right)^{2}}{\bar{b}}, \qquad (21) \\
\hat{y}_{t} &= \frac{-0.5(1-\rho_{y})\sigma_{y}^{2}}{(1-\rho_{y}^{2})} + \rho_{y} \hat{y}_{t-1} + \sigma_{y} e_{y,t}, \qquad (22) \\
\hat{a}_{t} &= \frac{-0.5(1-\rho_{a})\sigma_{a}^{2}}{(1-\rho_{a}^{2})} + \rho_{a} \hat{a}_{t-1} + \sigma_{a} e_{a,t} \qquad (23) \\
\hat{q}_{t} &= \gamma\left(1-\rho_{a}\right) \hat{a}_{t}. \qquad (24)
\end{aligned}
$$

Since the analytically tractable second-order approximate solution is not available, we use numerical approximations for quadratic solutions. We solve the five equation system above using Matlab algorithm `gensys2.m`. In actual simulations, we use "pruning" algorithm to avoid accumulation of high-order terms that can make the model unstable (Kim, Kim, Schaumburg, and Sims 2003). This pruning algorithm always produces stationary second-order accurate dynamics whenever the first-order dynamics are stable.

3.3 Comparison: Bond holding costs

A by-product of the utility modification method is that the model becomes stationary and the steady state bond holdings can be uniquely determined (zero in this model). Another popular way to make the model stationary is to adopt bond holding adjustment costs.[9] The process of achieving stationarity through modification of utility function works exactly same way as the bond holding cost does. If we introduce bond holding adjustment costs to the budget constraint, the budget constraint and Euler equation become

$$
\begin{aligned}
c_{t}^{i} + q_{t} b_{t}^{i} + \frac{\zeta}{2}\left(b_{t}^{i}\right)^{2} &= y_{t}^{i} a_{t} + b_{t-1}^{i}. \qquad (25) \\
\left(q_{t} + \zeta b_{t}\right)\left(c_{t}^{i}\right)^{-\gamma} &= \beta \mathrm{E}_{t}\left(c_{t+1}^{i}\right)^{-\gamma} \qquad (26)
\end{aligned}
$$

[9] See Kim and Kose (2003) and Schmitt-Grohe and Uribe (2003) for this and alternative methods to make incomplete market models stationary.

where $\frac{\zeta}{2}\left(b_t^i\right)^2$ represents quadratic bond holding adjustment costs.

In fact, the first order approximation of the model with bond holding costs is exactly same as that of the model with utility modification. This is why we use the same notation ζ for the constant parameter in the above equations. Even though the introduction of bond holding costs is not directly related to enforcing borrowing constraints, the behavior of the solution is quite similar in both models when we solve the model with second-order perturbation method as described in the previous section.[10] Therefore, it is worth while to compare the results of utility modification with those of bond holding costs, especially considering that bond holding costs are widely used in the incomplete market models. The simulation results which are not reported in this paper confirm that both methods produce similar results for impulse responses and second moments.

4 Model Performance

Parameter values that we use for simulation are summarized in the following table. The size of the idiosyncratic shock is estimated from the PSID data and quite large ($\sigma_y = 0.21$) compared to that is used in macro models (normally around 0.01). Implication of shock volatility on accuracy of the approximate solution is discussed later in this paper.

Parameter	Values
β	0.965
γ	1, 5
$\bar{b}$	0.2, 1
ρ_y	0.49
ρ_a	0.91
σ_y	0.21
σ_a	0.01
ζ	various values

4.1 Impulse responses

One advantage of the `gensys2.m` algorithm is that one does not have to distinguish state and control variables in solving the model. In particular, this "state-free" approach becomes useful when one solves complicated

[10] In fact, bond holding costs work in a symmetric way by reducing volatility of bond holdings so that there is less chance that borrowing limit is violated, while utility modification works in an asymmetric way that reflects inequality constraint.

models with a large number of state variables. However, when one wants to have decomposition into traditional state and control variables (for example, deriving so-called policy function), `gensys2.m` does not readily provide this decomposition. Kim, Kim, Schaumburg and Sims (2003) explains a way to achieve this decomposition using another Matlab algorithm called `gstate.m`.[11]

In this section, we report impulse responses of key variables to various shocks to the economy. One can easily generate impulse responses from policy functions by plugging in actual value of shocks. Readers can compare the policy functions of this and other algorithms by comparing impulse responses. We assume that the economy starts from the deterministic steady state (bond holdings of the representative agent at the steady state are set to zero).

4.1.1 Aggregate shock

In tables 1 and 2, we report impulse responses of consumption, bond holding and bond price (q_t) to one standard deviation shock (positive and negative) to aggregate and idiosyncratic endowments. We compare solutions of linear and quadratic approximations. We experiment with three values for ζ (0, 0.001 and 0.1), two values for γ (1 and 5), and two values of $\bar{b}$ (0.2 and 1).

Table 1 presents the impulse responses to positive and negative aggregate shocks (1 percent) to endowment a_t. Since all agents receive same shocks, there is no bond trading and the economy simply behaves like autarky. Impulse responses reflect this behavior and show that bond holdings stay at zero all the time. Both linear and quadratic approximations produce exactly same results because asset pricing rule is linear (quadratic approximation of a linear function is still linear) and consumption is a linear function of asset price. Since bond holdings stay at zero, consumption moves exactly same as output (as in autarky). Asset price increases with positive shocks because increased endowment creates excess supply that pushes interest rate down (asset price up). Note that asset price is a positive function of aggregate endowment as shown in equation (15). The table also shows that changes in γ affects bond prices only: q_t becomes more sensitive to endowment shocks when γ increases. Since bond holdings do not change, the model produce the same impulse responses irrespective of the values of parameters ζ and $\bar{b}$.

[11] All Matlab and R algorithms are publicly available at http://sims.princeton.edu/yftp/gensys2.

4.1.2 Idiosyncratic shock

Table 2 presents the impulse responses to positive and negative shocks to idiosyncratic components of endowment y_t. The first panel in table 2 reports the case of a positive shock; individual endowment increases by 21 percent in the first period and eventually dies out in 30 periods with persistence parameter of 0.49. Since asset price q_t is a function of aggregate shock only, q_t does not respond to the idiosyncratic shock.

Results show that impulse responses present the standard characteristics of typical incomplete markets models as shown in Mendoza (1991) and Baxter and Crucini (1995). With a positive shock, agents save a part of increased income by accumulating bonds over time. When $\zeta = 0$ (no utility modification), there is a permanent effect of temporary shock on consumption (nonstationary property of the incomplete market model). However,as ζ increases, both consumption and bond holding processes become stationary by reacting more to shocks at the initial period and eventually moving back to the original steady state. Changing the value of γ affects the impulse responses when ζ is positive but there is no effect when when $\zeta = 0$. We observe more consumption smoothing with higher γ. Comparison of linear and quadratic solutions shows that linear approximation underestimates the responses of consumption compared to the quadratic approximation. As ζ increases, differences between linear and quadratic solutions increase.

The second panel in table 2 presents the case with a negative shock. With a negative shock, agents borrow (minus b_t) to smooth out consumption over time. In the case of a linear solution, the response of consumption is exactly the opposite to that of the positive shock case. However, the existence of borrowing limits affect optimal behavior of bond holdings. The absolute amount of borrowing when agents face negative shock is less than the absolute amount of bond accumulation when agents face the same magnitude of positive shock. For example, with positive shock ($\gamma = 1, \zeta = 0$), bond holdings increase by 22% at impact, while with the same magnitude of negative shock, agents' borrowings increase only by 18%. This phenomenon becomes more significant when borrowing limit ($\bar{b}$) is decreased to 0.2. Panel 3 of table 2 shows how bond holdings react when $\bar{b} = 0.2$ compared to the case when $\bar{b} = 1$. With the same size of negative shock, agents do not borrow as much as in the case when $\bar{b} = 1$. For example, borrowing in the first period becomes only 13% instead of 18% when $\bar{b}$ decreases from 1 to 0.2 (with $\gamma = 1, \zeta = 0$). Consumption process remains unchanged with changes in $\bar{b}$.

4.2 Moments

In this section, we report main properties of simulated time series of variables when both aggregate and idiosyncratic shocks hit the economy. All the statistics are based on 500 simulations where each simulation lasts for 100 periods. Since the moments are sensitive to some parameter values, we experiment with three values for ζ (0.1, 0.001 and 10^{-10}), two values of γ (1 and 5), and two values of $\bar{b}$ (1 and 0.2).[12] Actual shocks are generated by random number generator in Matlab with the characteristics reported in the model parameter table. We report correlation between consumption and endowment shocks (aggregate and idiosyncratic), and between consumption and bond holdings. We also report standard deviation and autocorrelation parameters (up to three lags) of consumption, bond holdings and asset price. We report statistics generated by linear and quadratic solutions.

As in the previous section, the statistics are consistent with standard properties of incomplete market models. Consumption is procyclical and less volatile than output (standard deviations of aggregate and idiosyncratic shocks are 0.01 and 0.21, respectively). Standard deviation of consumption is around $5\% \sim 15\%$ depending mostly on the value of ζ. When ζ increases, consumption volatility increases because consumption moves more similarly to the changes in output (less degree of consumption smoothing). Correlation between consumption and output shocks are positive around $0.2 \sim 0.8$, depending on the parameter values of γ and ζ. Consumption and bond holdings are also positively correlated. As ζ increases, consumption correlation with idiosyncratic shock and bond holdings increase because of less amount of consumption smoothing. When ζ is low, consumption process becomes near nonstationary while output process remains stationary, which lowers correlation between consumption and idiosyncratic shock. On the other hand, correlation between consumption and aggregate shock decreases as ζ increases. Both consumption and bond holding processes are quite persistent with autocorrelation parameter with lag 1 are around $0.7 \sim 0.9$. Changes in $\bar{b}$ does not affect consumption process but they affect bond holdings. Bond holdings become less persistent and less correlated with consumption with tighter borrowing constraint (low $\bar{b}$).

Bond price q_t is quite persistent with autocorrelation parameter is above 0.85 with lag 1. Process of q_t does not depend on any parameters (ζ nor $\bar{b}$) except that the volatility of bond price is affected by γ; when γ increases from 1 to 5, standard deviation of q_t decreases to a half. Comparing linear

[12] We so not use zero for ζ because when ζ is zero, second moments are not well defined due to nonstationarity.

and quadratic solutions reveal that both solutions produce similar statistics except that linear solution slightly underestimates consumption volatility. Since bond price q_t is a linear process, both linear and quadratic solutions produce same statistics for q_t.

4.3 Wealth distribution

In this section, we analyze wealth distribution predicted by the model using simulated series of bond holdings. We simulate the economy 1000 times with 100 periods in each simulation. We experiment with two values of $\bar{b}$ (0.2 and 1) while other parameter values are set at $\gamma = 1, \zeta = 0.001$. We only use quadratic solution for this exercise. All simulations start from the deterministic steady state with zero bond holdings.

Since we incorporate borrowing constraint in the way that borrowing constraints never bind, bond holdings never hit the limit. However, bond holdings can reach very close to the limit. We derive the fraction of times that agents are constrained by borrowing limits by approximating how often b_t moves into a certain range above the borrowing limit. Table 4 reports the results. When $\bar{b}$ is 1, bond holdings move within 10^{-5} of borrowing limit (it is -0.99999) only less than 1% of times. About 44% of times, agents end up in a borrowing situation. In 0.02% of times, agents accumulate bond holdings over 10, which means that agents' net assets become ten times more than the steady state level of consumption..

However, with a tighter borrowing constraint $\bar{b} = 0.2$, agents hit the borrowing constraint around 50% of times (when using the same definition of 10^{-5} range). Wealth distribution has a thicker right tail when $\bar{b} = 0.2$ as agents accumulate bonds over 10 in about 8% of times.

4.4 Accuracy Tests

We evaluate the accuracy of linear and quadratic solutions using Euler equation errors. In order to deal with the possible accumulation of Euler equation errors, we proceed as follows. First, we create 50 hypothetical agents that are not part of the economy but have the realization of shocks that are identical to those of the first 50 agents of the sample. Their consumption decision, however, is not calculated with the numerical solution but is calculated explicitly from the Euler equation. Then, we calculate percentage differences of this consumption series from the actual consumption series from our quadratic solution.

In calculating consumption series based on the Euler equation, we take conditional expectation of c_{t+1} in Euler equation in time t using two-dimensional Gauss-Hermite quadrature with ten nodes.[13] Note that c_{t+1} is a function of bond holdings and that future bond holdings are calculated based on the quadratic solution. Since the original utility function does not contain penalty terms, it would be appropriate to use the original Euler equation without utility modification when calculating approximation errors.

Table 5 reports test results. Each simulation is based on 500 periods and mean and maximum absolute errors are reported. Errors are expressed as percentage of steady state consumption. We report errors in some periods [10, 20, 50, 100, 200, 500] in order to detect possible accumulation of errors. We experiment with two values for borrowing limit $\bar{b}$. The value of ζ is set to minimize approximation errors for each solution method and each set of parameter values.

The first panel is the case when $\bar{b} = 1$. Approximation errors are minimized when $\zeta = 0.05$. Euler equation errors are between 1.3% and 1.8% of steady state consumption in both linear and quadratic approximations. The quadratic solution generates slightly smaller approximation errors than the linear solution. With $\bar{b} = 0.2$, approximation errors increase to $4 \sim 5.5$ percent of steady state consumption. With tighter borrowing constraints (that is, smaller $\bar{b}$), the relative performance of quadratic approximation over linear approximation improves. The mean errors of quadratic approximation are as much as one percentage point smaller than those of linear approximation, while the difference was between $0.2 \sim 0.5\%$ with $\bar{b} = 1$.

Absolute level of approximation errors can be quite large compared to that of other nonlinear approximation methods. This is because the accuracy of perturbation method directly related to the volatility of underlying shocks of the economy. The current specification has highly volatile idiosyncratic shocks (standard deviation of 21%). If we lower the shock variance, then the approximation errors would significantly go down.[14]

In Figure 1, we rearrange average Euler equation errors conditional on the level of bond holdings at the start of the period. The graphs show that average errors dramatically increase as bond holdings approach the borrowing limit, in particular when $\bar{b} = 0.2$. In the original Kuhn-Tucker condition, Euler equation does not have to hold when borrowing hits the limit. The graphs also show that the errors increase with high level of initial bond holdings. This is natural because perturbation method is valid only

[13] See Judd (1998) for details.

[14] We experimented with $\sigma_y = 0.021$ (one tenth of the original σ). Approximation errors decrease to around 0.1% (from around $1.3 \sim 1.8\%$).

locally around the steady state (which is zero in this model) and therefore approximation errors would increase as the economy moves away from the steady state.

5 Conclusion

This paper explains how to use a perturbation method to solve an incomplete market model with infinite number of agents and exogenous borrowing constraints. Traditionally, perturbation method has not been used to solve this type of model because perturbation method is a local approximation and deals mostly with equality constraints. We adopt the barrier method to convert the maximization problem with inequality constraints into an equality constraint problem. Simulation results show that the perturbation solution generates quite reasonable results in terms of second moments and impulse responses. Accuracy of the solution can be a problem especially when underlying shocks become large. However, considering the computation time and the compactness of solution, perturbation methods deserve some consideration compared to other computationally intensive projection solution methods.

A Derivation of the Linear Solution

The linearized version of the first order conditions (budget constraint and Euler equation) for each agent (country) i can be expressed in the following linear system (expectation operator is omitted for convenience),

$$\begin{bmatrix} 1 & \beta \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \gamma & -\zeta \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} \hat{y}_t^i + \hat{a}_t \\ -\hat{q}_{t-1} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & \beta \\ \gamma & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\beta} & -\frac{1}{\beta\gamma} \end{bmatrix}$$

the system becomes

$$\begin{aligned} \begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\beta} & -\frac{1}{\beta\gamma} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \gamma & -\zeta \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\beta} & -\frac{1}{\beta\gamma} \end{bmatrix} \begin{bmatrix} \hat{y}_t^i + \hat{a}_t \\ -\hat{q}_{t-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{\zeta}{\gamma} \\ -\frac{1}{\beta} & \frac{1}{\beta}\left(1 + \frac{\zeta}{\gamma}\right) \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} -\frac{\hat{q}_{t-1}}{\gamma} \\ \frac{1}{\beta}\left(\hat{y}_t^i + \hat{a}_t + \frac{\hat{q}_{t-1}}{\gamma}\right) \end{bmatrix} \end{aligned}$$

Based on the Jordan decomposition

$$\begin{aligned} &\begin{bmatrix} 1 & -\frac{\zeta}{\gamma} \\ -\frac{1}{\beta} & \frac{1}{\beta}\left(1 + \frac{\zeta}{\gamma}\right) \end{bmatrix} \\ = &\begin{bmatrix} \left(\Delta + \frac{\zeta}{\gamma}\right) & -\frac{\beta}{1-\beta}\frac{\zeta}{\gamma} \\ 1 & \frac{\Delta}{(1-\beta)\lambda} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda\beta} \end{bmatrix} \begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \lambda &= \frac{1}{2}\left[\left(1 + \beta^{-1} + \beta^{-1}\gamma^{-1}\zeta\right) - \sqrt{\left(1 + \beta^{-1} + \beta^{-1}\gamma^{-1}\zeta\right)^2 - 4\beta^{-1}}\right], \\ \Delta &= 1 - \lambda\beta \end{aligned}$$

we convert the system

$$
\begin{aligned}
&\begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} \\
= &\begin{bmatrix} \left(\Delta + \frac{\zeta}{\gamma}\right) & -\frac{\beta}{1-\beta}\frac{\zeta}{\gamma} \\ 1 & \frac{\Delta}{(1-\beta)\lambda} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda\beta} \end{bmatrix} \begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} \\
&+ \begin{bmatrix} -\frac{\hat{q}_{t-1}}{\gamma} \\ \frac{1}{\beta}\left(\hat{y}_t^i + \hat{a}_t + \frac{\hat{q}_{t-1}}{\gamma}\right) \end{bmatrix}
\end{aligned}
$$

that is

$$
\begin{aligned}
&\begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} \\
= &\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} \\
&+ \begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} -\frac{\hat{q}_{t-1}}{\gamma} \\ \frac{1}{\beta}\left(\hat{y}_t^i + \hat{a}_t + \frac{\hat{q}_{t-1}}{\gamma}\right) \end{bmatrix}
\end{aligned}
$$

To suppress the diverging eigenvalue, we should have

$$
-\hat{c}_t^i + \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\hat{b}_t^i = -\sum_{s=0}^{\infty}\beta^s \mathrm{E}_t\left[\frac{\hat{q}_{t+s}}{\gamma} + \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right)\right].
$$

Therefore, the solution for agent i given exogenous interest rate becomes

$$
\begin{aligned}
\frac{\hat{b}_t^i}{\lambda} &= \hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t - \sum_{s=0}^{\infty}\beta^s \mathrm{E}_t\left[(1-\beta)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right) + \frac{\hat{q}_{t+s}}{\gamma}\right] \\
\frac{\hat{c}_t^i}{\lambda} &= \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\left(\hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t\right) \\
&\quad + \sum_{s=0}^{\infty}\beta^{s+1}\mathrm{E}_t\left[(1-\beta)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right) + \frac{\hat{q}_{t+s}}{\gamma}\right]
\end{aligned}
$$

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Table 1. Impulse responses to aggregate shock

γ=1 (ζ=0, 0.001, 0.1 and bbar=0.2, 1)

	positive shock (1%)			negative shock (-1%)		
period	c_{t}	b_{t}	q_{t}	c_{t}	b_{t}	q_{t}
1	0.01	0	0.0009	-0.01	0	-0.0009
2	0.0091	0	0.0008	-0.0091	0	-0.0008
3	0.0083	0	0.0007	-0.0083	0	-0.0007
10	0.0043	0	0.0004	-0.0043	0	-0.0004
30	0.0007	0	0.0001	-0.0007	0	-0.0001

γ=5 (ζ=0, 0.001, 0.1 and bbar=0.2, 1)

	positive shock (1%)			negative shock (-1%)		
period	c_{t}	b_{t}	q_{t}	c_{t}	b_{t}	q_{t}
1	0.01	0	0.0045	-0.01	0	-0.0045
2	0.0091	0	0.0041	-0.0091	0	-0.0041
3	0.0083	0	0.0037	-0.0083	0	-0.0037
10	0.0043	0	0.0019	-0.0043	0	-0.0019
30	0.0007	0	0.0003	-0.0007	0	-0.0003

Table 2. Impulse responses to idiosyncratic shock (one standard deviation)

Positive shock (21%) with bbar=1

	y_{t}	c_{t}		b_{t}		q_{t}
		linear	quad	linear	quad	
(γ=1,ζ=0)						
1	0.21	0.0139	0.0149	0.2253	0.2267	0
2	0.1029	0.0139	0.0149	0.3535	0.3284	0
3	0.0504	0.0139	0.0149	0.4212	0.3744	0
10	0.0003	0.0139	0.0149	0.4889	0.4156	0
30	0	0.0139	0.0149	0.4894	0.4158	0
(γ=1,ζ=0.001)						
1	0.21	0.0209	0.0209	0.2165	0.2206	0
2	0.1029	0.0207	0.0207	0.3341	0.317	0
3	0.0504	0.0204	0.0205	0.3907	0.3582	0
10	0.0003	0.0179	0.0186	0.3896	0.3637	0
30	0	0.0124	0.0138	0.2567	0.27	0
(γ=1,ζ=0.1)						
1	0.21	0.0932	0.0968	0.1286	0.137	0
2	0.1029	0.0807	0.0832	0.1599	0.1644	0
3	0.0504	0.0653	0.0674	0.1483	0.1515	0
10	0.0003	0.009	0.0096	0.0229	0.0245	0
30	0	0	0	0.0001	0.0001	0
(γ=5,ζ=0.001)						
1	0.21	0.0159	0.0164	0.2229	0.2252	0
2	0.1029	0.0158	0.0164	0.3482	0.3255	0
3	0.0504	0.0157	0.0163	0.4127	0.3703	0
10	0.0003	0.0152	0.0159	0.4595	0.4027	0
30	0	0.0138	0.0148	0.4092	0.3776	0
(γ=5,ζ=0.1)						
1	0.21	0.0544	0.0579	0.175	0.1805	0
2	0.1029	0.051	0.0538	0.2471	0.2411	0
3	0.0504	0.0465	0.0487	0.2623	0.251	0
10	0.0003	0.0197	0.0208	0.1211	0.1224	0
30	0	0.0017	0.0018	0.01	0.0107	0

Negative shock (-21%) with bbar=1

	y_{t}	c_{t}		b_{t}		q_{t}
		linear	quad	linear	quad	
(γ=1,ζ=0)						
1	-0.21	-0.0139	-0.013	-0.1839	-0.1829	0
2	-0.1029	-0.0139	-0.013	-0.2612	-0.2749	0
3	-0.0504	-0.0139	-0.013	-0.2964	-0.3195	0
10	-0.0003	-0.0139	-0.013	-0.3284	-0.3615	0
30	0	-0.0139	-0.013	-0.3286	-0.3617	0
(γ=1,ζ=0.001)						
1	-0.21	-0.0209	-0.0209	-0.1779	-0.1751	0
2	-0.1029	-0.0207	-0.0207	-0.2504	-0.26	0
3	-0.0504	-0.0204	-0.0203	-0.2809	-0.2977	0
10	-0.0003	-0.0179	-0.0173	-0.2803	-0.2937	0
30	0	-0.0124	-0.0111	-0.2043	-0.1959	0
(γ=1,ζ=0.1)						
1	-0.21	-0.0932	-0.0897	-0.114	-0.1074	0
2	-0.1029	-0.0807	-0.0783	-0.1379	-0.1346	0
3	-0.0504	-0.0653	-0.0633	-0.1292	-0.1268	0
10	-0.0003	-0.009	-0.0083	-0.0224	-0.0208	0
30	0	0	0	-0.0001	-0.0001	0
(γ=5,ζ=0.001)						
1	-0.21	-0.0159	-0.0153	-0.1822	-0.1807	0
2	-0.1029	-0.0158	-0.0152	-0.2582	-0.2707	0
3	-0.0504	-0.0157	-0.0152	-0.2921	-0.3134	0
10	-0.0003	-0.0152	-0.0145	-0.3149	-0.3415	0
30	0	-0.0138	-0.0128	-0.2904	-0.3063	0
(γ=5,ζ=0.1)						
1	-0.21	-0.0544	-0.0509	-0.1489	-0.1449	0
2	-0.1029	-0.051	-0.0483	-0.1982	-0.202	0
3	-0.0504	-0.0465	-0.0442	-0.2078	-0.2149	0
10	-0.0003	-0.0197	-0.0186	-0.108	-0.107	0
30	0	-0.0017	-0.0016	-0.0099	-0.0093	0

Changes in b_{t} when bbar=0.2

(only b_{t} changes when bbar changes)

	b_{t} (positive shock)		b_{t} (negative shock)	
	linear	quad	linear	quad
(γ=1,ζ=0)				
1	0.3523	0.1677	-0.1276	-0.1518
2	0.7086	0.1308	-0.156	-0.184
3	0.9596	0.0851	-0.1655	-0.1915
10	1.2635	0.0331	-0.1727	-0.1956
30	1.2658	0.0327	-0.1727	-0.1957
(γ=1,ζ=0.001)				
1	0.3328	0.1805	-0.1249	-0.1464
2	0.6451	0.1669	-0.1527	-0.1795
3	0.8403	0.1411	-0.1616	-0.1874
10	0.8361	0.2153	-0.1614	-0.1845
30	0.427	0.4163	-0.1362	-0.1373
(γ=1,ζ=0.1)				
1	0.1662	0.1534	-0.0908	-0.0946
2	0.22	0.19	-0.1048	-0.1115
3	0.1994	0.188	-0.0998	-0.1027
10	0.024	0.0338	-0.0214	-0.0136
30	0.0001	0.0001	-0.0001	0
(γ=5,ζ=0.001)				
1	0.3469	0.1727	-0.1269	-0.1502
2	0.6907	0.1435	-0.1551	-0.1827
3	0.9253	0.1038	-0.1645	-0.1904
10	1.1247	0.0929	-0.1698	-0.1933
30	0.9115	0.2138	-0.164	-0.1866
(γ=5,ζ=0.1)				
1	0.2479	0.1809	-0.1107	-0.1241
2	0.4034	0.217	-0.1337	-0.1542
3	0.441	0.2327	-0.1376	-0.1579
10	0.1542	0.1901	-0.0871	-0.0756
30	0.0103	0.0166	-0.0098	-0.004

Table 3. Moments

bbar	1						0.2					
γ	1			5			1			5		
ζ	10^(-10)	0.001	0.1	10^(-10)	0.001	0.1	10^(-10)	0.001	0.1	10^(-10)	0.001	0.1
linear												
corr(C,A)	0.3437	0.2951	0.1421	0.3437	0.3302	0.1867	0.3437	0.2951	0.1421	0.3437	0.3302	0.1867
corr(C,Y)	0.2241	0.373	0.8531	0.2241	0.2707	0.69	0.2241	0.373	0.8531	0.2241	0.2707	0.69
corr(C,B)	0.7588	0.8214	0.9578	0.7588	0.781	0.9251	0.4156	0.4637	0.7861	0.4156	0.4327	0.6148
s.d.(C)	0.0579	0.0665	0.1558	0.0579	0.0595	0.1141	0.0579	0.0665	0.1558	0.0579	0.0595	0.1141
autocorr(C)-1	0.9184	0.9076	0.7712	0.9184	0.9152	0.8518	0.9184	0.9076	0.7712	0.9184	0.9152	0.8518
-2	0.8434	0.8217	0.5661	0.8434	0.8371	0.7126	0.8434	0.8217	0.5661	0.8434	0.8371	0.7126
-3	0.7735	0.742	0.4013	0.7735	0.7643	0.5889	0.7735	0.742	0.4013	0.7735	0.7643	0.5889
autocorr(B)-1	0.9261	0.9323	0.8681	0.9261	0.9294	0.9164	0.6543	0.6807	0.7561	0.6543	0.6666	0.728
-2	0.8279	0.8322	0.6726	0.8279	0.8313	0.7829	0.3843	0.4063	0.4865	0.3843	0.3935	0.4505
-3	0.7287	0.728	0.4876	0.7287	0.7308	0.6447	0.2296	0.2418	0.2948	0.2296	0.2358	0.2716
mean(q)	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654
s.d.(q)	0.0018	0.0018	0.0018	0.009	0.009	0.009	0.0018	0.0018	0.0018	0.009	0.009	0.009
autocorr(q)-1	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567
-2	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327
-3	0.6231	0.6231	0.6231	0.623	0.623	0.623	0.6231	0.6231	0.6231	0.623	0.623	0.623
quad												
corr(C,A)	0.3215	0.2876	0.1417	0.3329	0.3169	0.1858	0.3215	0.2876	0.1417	0.3329	0.3169	0.1858
corr(C,Y)	0.2149	0.3702	0.8592	0.2104	0.2577	0.6958	0.2149	0.3702	0.8592	0.2104	0.2577	0.6958
corr(C,B)	0.3555	0.5039	0.961	0.3395	0.3843	0.9287	0.0834	0.1022	0.5935	0.073	0.0802	0.2812
s.d.(C)	0.0676	0.0717	0.1586	0.069	0.0677	0.1128	0.0676	0.0717	0.1586	0.069	0.0677	0.1128
autocorr(C)-1	0.9217	0.9083	0.7681	0.922	0.9186	0.8509	0.9217	0.9083	0.7681	0.922	0.9186	0.8509
-2	0.8498	0.8237	0.561	0.8503	0.8437	0.7105	0.8498	0.8237	0.561	0.8503	0.8437	0.7105
-3	0.7838	0.7455	0.3957	0.7844	0.7747	0.5857	0.7838	0.7455	0.3957	0.7844	0.7747	0.5857
autocorr(B)-1	0.9275	0.9357	0.8682	0.9275	0.9306	0.9196	0.5781	0.6376	0.7264	0.5786	0.6039	0.6895
-2	0.8312	0.8398	0.6734	0.831	0.8355	0.7898	0.3188	0.3822	0.4405	0.3172	0.3398	0.4119
-3	0.7368	0.7398	0.4892	0.7366	0.7413	0.6537	0.2073	0.254	0.2588	0.2058	0.2246	0.256
mean(q)	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654
s.d.(q)	0.0018	0.0018	0.0018	0.009	0.009	0.009	0.0018	0.0018	0.0018	0.009	0.009	0.009
autocorr(q)-1	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567
-2	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327
-3	0.6231	0.6231	0.6231	0.623	0.623	0.623	0.6231	0.6231	0.6231	0.623	0.623	0.623

Table 4. Wealth Distribution

	bbar=1	bbar=0.2
	fraction	fraction
within 10^(-6)	0.41%	46.52%
within 10^(-5)	0.83%	49.57%
within 10^(-4)	1.75%	53.20%
within 10^(-3)	3.60%	57.45%
within 10^(-2)	7.35%	62.65%
within 0.1	16.07%	69.71%
b<0	43.39%	72.72%
b<5	98.47%	89.36%
b<10	99.98%	91.94%

percentile	b	b
5%	-0.997	-0.2
10%	-0.9747	-0.2
90%	2.8933	5.9014
95%	2.8994	5.9749

Table 5. Accuracy tests (Euler equation errors)
(50 agents with 500 period each)

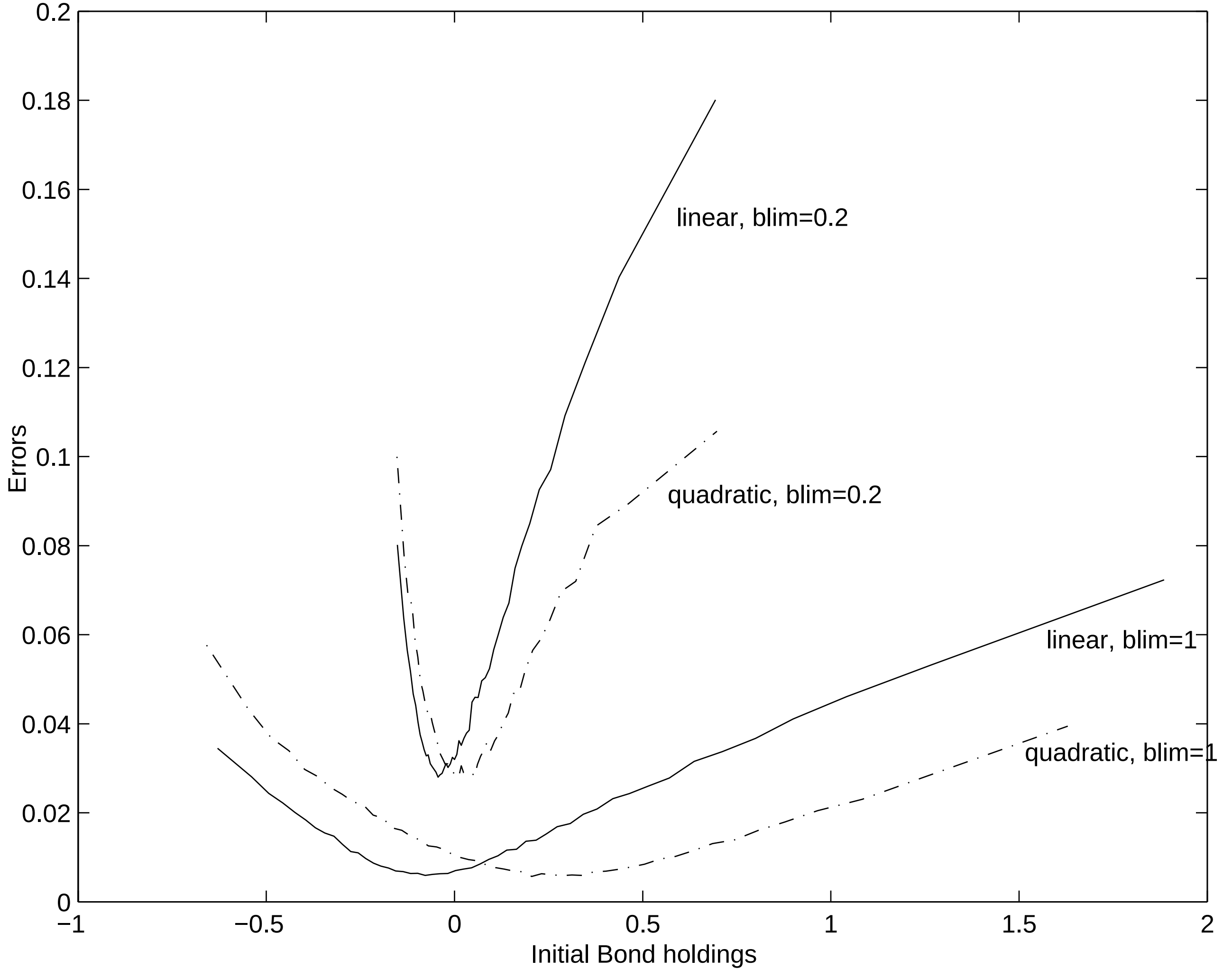
bbar=1 ζ=0.05

	Linear		**Quad**	
period	mean	*max*	mean	*max*
10	1.66%	*6.40%*	1.70%	*5.64%*
20	1.64%	*4.08%*	1.63%	*5.59%*
30	1.81%	*10.90%*	1.66%	*7.66%*
50	1.28%	*4.28%*	1.38%	*5.22%*
100	1.87%	*8.08%*	1.29%	*5.09%*
200	1.64%	*8.45%*	1.35%	*5.25%*
500	1.82%	*10.11%*	1.57%	*6.46%*

bbar=0.2 ζ=0.5

	Linear		**Quad**	
period	mean	*max*	mean	*max*
10	4.66%	*20.42%*	4.17%	*13.55%*
20	4.53%	*15.00%*	4.10%	*11.36%*
30	5.00%	*19.62%*	4.66%	*16.21%*
50	3.83%	*11.18%*	3.42%	*7.35%*
100	5.03%	*18.41%*	4.09%	*12.03%*
200	5.21%	*28.38%*	4.37%	*14.73%*
500	5.59%	*40.98%*	4.77%	*19.15%*

Figure 1. Mean Euler equation errors conditional on initial bond holdings



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