The idea of using generating functions to solve recurrence relations is to formally define a power series $f(x) = \sum a_k x^k$ whose coefficients obey the recurrence relationship. The recurrence relationship then gives a functional equation for $f(x)$, which can be solved to express $f(x)$ in closed form. The power series expansion for $f(x)$ can then be used to find closed-form expressions for $a_k$.

**Example**. Let $a_0 = 0$ and $a_1 = 1$, Suppose $a_{k+2} = a_{k+1} + a_k$. Find a closed-form expression for $a_k$. (Note: this is mathematically equivalent to the rabbit problem: A single pair of rabbits (generation 0) is released into an uninhabited forest. After one month, they do what rabbits do best, and another pair of rabbits (generation 1) is born. At the end of each subsequent month, a new pair of rabbits is born for each pair of rabbits that were born in the past two months (so that 2 pairs are born at the end of the second month, 3 at the end of the third, 5 at the end of the fourth, and so on. How many rabbits are born in the $k$th generation?

**Solution**. Consider the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Multiplying both sides of the recurrence relation by $x^k$ and summing over k gives

$$\sum_{k=0}^{\infty} a_{k+2} x^k = \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k. \tag{1}$$

Notice that

$$x \sum_{k=0}^{\infty} a_{k+1} x^k = \sum_{k=0}^{\infty} a_{k+1} x^{k+1} = \sum_{k=1}^{\infty} a_k x^k = f(x) - a_0$$

and, similarly,

$$x^2 \sum_{k=0}^{\infty} a_{k+2} x^k = \sum_{k=0}^{\infty} a_{k+2} x^{k+2} = \sum_{k=2}^{\infty} a_k x^k = f(x) - a_0 - a_1 x.$$

Thus, we can rewrite Equation (1) as

$$\frac{1}{x^2}\left(f(x) - a_0 - a_1 x\right) = \frac{1}{x}\left(f(x) - a_0\right) + f(x),$$

or,

$$(1 - x - x^2) f(x) = a_0 + (a_1 - a_0)x = x.$$

So, $f(x) = x/(1 - x - x^2)$. Factoring $1 - x - x^2 = (1 - \varphi x)(1 - \widehat{\varphi} x)$ and using partial fractions, we can then write

$$f(x) = \frac{1/\sqrt{5}}{1 - \varphi x} - \frac{1/\sqrt{5}}{1 - \widehat{\varphi} x},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\widehat{\varphi} = \frac{1-\sqrt{5}}{2}$. The power series for

$$\frac{1}{1 - \varphi x} = \sum_{k=0}^{\infty} \varphi^k x^k$$

with a similar expression for $\widehat{\varphi}$, giving

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{5}}\left(\varphi^k - \widehat{\varphi}^k\right)\right) x^k,$$

thus, we have $a_k = \frac{1}{\sqrt{5}}\left(\varphi^k - \widehat{\varphi}^k\right)$.

**Example**. Let $a_0 = 0$ and $a_1 = 2$, Suppose $a_{k+2} = 4a_{k+1} - 4a_k$. Find a closed-form expression for $a_k$.

**Solution**. As before, we multiply both sides of the recurrence relation by $x^k$ and sum over $k$. Doing so, and substituting in for $a_0$ and $a_1$ gives

$$f(x) = \frac{2x}{(1-2x)^2}.$$

(Check this!) We can't directly write down a power series for $f(x)$, however, due to the squared term in the denominator. Instead, notice that $f(x) = xg'(x)$, where $g(x) = \frac{1}{1-2x}$. Thus, we can write the power series for $g(x)$ as

$$g(x) = \sum_{k=0}^{\infty} 2^k x^k,$$

So the power series for $f(x)$ is

$$f(x) = x\sum_{k=0}^{\infty} k2^k x^{k-1} = \sum_{k=1}^{\infty} k2^k x^k,$$

so $a_k = k2^k$.

**Warm-Up Exercises**.

1. Let $a_0 = 2$, $a_1 = 5$, and $a_{k+2} = 5a_{k+1} - 6a_k$. Find $a_k$.
2. Let $a_0 = 6$, $a_{k+1} = 2a_k + 2(k+1)$. Find $a_k$.
3. Solve the recurrence $(k+1)(k+2)a_{k+2} - 3(k+1)a_{k+1} + 2a_k = 0$, with $a_0 = 2$, $a_1 = 3$.
4. (a) The *Bernoulli Numbers* $b_k$ can be defined using an exponential generating function, $B(x) = \sum_k \frac{b_k x^k}{k!} = x/(e^x - 1)$. Use this to compute $b_0$ through $b_3$.
   (b) Show that $\sum_{k=0}^{m} \binom{m+1}{k} b_k = 0$ for $m > 0$. (Note that for $m = 0$, this can be directly computed to be 1.) Hint: Find the power series representation of $B(x)(e^x - 1)$.
   (c) Let $S_m(n) = \sum_{k=1}^{n-1} k^m$. Show $S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} b_k n^{m+1-k}$.
5. Let $a_0 = 1$, and $a_k = 7 + \frac{1}{a_{k-1}}$. Find $a_k$.

**Putnam Problems**.

1. (Putnam 1939) Given the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$, with $a_k = (k^2+1)3^k$, show that there is a recurrence relation of the form $a_k + pa_{k+1} + qa_{k+2} + ra_{k+3} = 0$, where $p$, $q$, and $r$ are independent of $k$. Find $p$, $q$, and $r$, as well as $f(x)$.
2. (Putnam 1999, A-6) The sequence $a_k$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and
   $$a_k = \frac{6a_{k-1}^2 a_{k-3} - 8a_{k-1}a_{k-2}^2}{a_{k-2}a_{k-3}}.$$
   Show that, for all $k$, $a_k$ is a multiple of $k$.
3. (Putnam 2005, B-4) For positive integers $m$ and $n$, let $f(m,n)$ denote the number of $n$-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \ldots |x_n| \leq m$. Show that $f(n,m) = f(m,n)$.