Putnam Sessions **The Pigeonhole Principle** **Fall 2008**
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The pigeonhole principle states that if $n+1$ pigeons must go into n holes, then at least two pigeons must go into the same hole. This self-evident principle is useful in the solution of many problems. Usually the difficulty is to recognize who are the pigeons and what constitutes a hole.

Example. In a hotly contested election year, each senator slaps the face of one other senator (just one). Observing senatorial decorum, no two senators slap each other. Irrespective of how the senators slap one another, is it always possible to form a committee of 35 senators none of whom has slapped another committee member?

Solution. Represent each senator by a vertex. If senator A slaps B, draw an arrow from A to B. If A slaps B, B slaps C, and C slaps A, then the three senators A, B, C form a triangle. Suppose the other 97 senators all slap A. Then we can form a committee consisting of these 97 senators none of whom has slapped another committee member.

On the other hand, there is a configuration of slapping that does not allow the formation of a 35-member committee. If one senator slaps another, we put both of them into the same hole. Because no two senators slap each other, if A slaps B, then B must slap someone else C. This means each hole must hold at least three senators. Among 100 senators, there can be at most 33 holes. One possibility is to have 32 holes with three senators each and one hole with four senators. Given such a configuration, if there is a 34-member committee none of whose members has slapped another committee member, then at least two must go into the same hole. Since no two members may be in a three-senator hole, two must be in the four-senator hole. If the committee has thirty-five members, then the last member has to go into a three-senator hole or a four-senator hole. In either case, one committee member has slapped another. So in this case it is not possible to form a committee of 35 members none of whom has slapped another committee member.

Warm-Up Exercise.

1. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart.

Pigeonhole Problems.

1. (Putnam 1978, A-1) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \dots, 100$. Prove that there must be two distinct integers in A whose sum is 104. (Actually 20 can be replaced by 19.)

2. (Putnam 1971, A-1) Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points. (*Hint*: If P and Q are lattice points, when is the midpoint $(P+Q)/2$ also a lattice point?)

3. (Putnam 1958, B-2) Given a set of $n+1$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.

4. (Putnam 1954, A-2) Consider any five points P_1, P_2, P_3, P_4, P_5 in the interior of a square S of side-length 1. Denote by d_{ij} the distance between the points P_i and P_j. Prove that at least one of the distances d_{ij} is less than $\sqrt{2}/2$.

5. (Putnam 1980, A-4) Prove that there exist integers a, b, c, not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$

6. Show that if there are n people at a party, then two of them know the same number of people (among those present).

7. Let X be any real number. Prove that among the numbers

$$X, 2X, \ldots, (n-1)X$$

there is one that differs from an integer by at most $1/n$.