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FROM UNKNOWN AMOUNTS TO REPRESENTING VARIABLES

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Our work in Early Algebra has been built on the ideas that (a) generalizing lies at the heart of algebraic reasoning; (b) arithmetical operations can be viewed as functions; and (c) students' spontaneous representations of mathematical relations can be nurtured to evolve into meaningful notation based on mathematical conventions. We have also assumed that mathematical learning is an inherently social activity, closely tied to the discussions and contexts where it takes place. We will illustrate how such cognitive and social ideas underpin the design and implementation of third grade algebraic-arithmetic tasks related to the operations of addition and subtraction.

In planning the lesson we will describe, we had several specific considerations in mind, among which were the following:

1. Additive comparisons tended to be somewhat challenging to third graders;
2. Although the amount in each box was unspecified, the difference was fixed; from earlier work, we knew that a difference quantity is an important component of additive structures;
3. We wondered whether and how children would express the idea that there could be many possible values for two compared amounts;
4. The problem would possibly provide an opportunity to introduce letters to stand for unknown amounts and for variables; and
5. The activity would best be promoted through intellectually playful dialogue in a setting where students were encouraged to express their ideas and show their insightfulness into the problem.

Our example is part of our second to fourth grade longitudinal investigation with the students of four classrooms in a public elementary school from a multi-cultural working-class community in Greater Boston. Descriptions of our class materials are available at www.earlyalgebra.terc.edu. The lesson we focus upon in this paper was the first one we taught to third grade students we had been working with since second grade.

The lesson is structured around a situation in which one child, Mary, is said to have three more candies than another child, John. We actually show two boxes, one Mary's, the other John's, and assert that there are equal amounts of candies in each. We then take an additional three candies and place them on top of Mary's box and state that her total amount includes both the candies inside and on top of the box. We begin by asking the students to express what they know about John and Mary's amounts.

The problem can be viewed both as a particular empirical state of affairs and as a set of logical possibilities. The former viewpoint gains prominence when one wonders how many candies are actually in the box. The latter emerges as one focuses on the verbal description of the problem and attempts to find multiple "solutions". Each viewpoint offers its version of truth or correctness. There can only be one empirical truth regarding the number of candies John and Mary have. By this standard, only students who guess the numbers of candies in the boxes can be right. However, the logical standard accepts all answers consistent with the information given, regardless of whether they correspond to the reality present in the classroom. As one student expressed it, "Everybody [in her class] had the right answer... Because everybody... has three more. Always." This second viewpoint supports the idea of a variable: there are multiple possibilities that would make the basic premise of the problem true.

By asking the students to make predictions about numbers of candies, we may have encouraged some of them to construe their task as having to guess accurately. However, this served as an opportunity to discuss "impossible answers", such as the suggestion that one child had 8 candies and the other 10 candies. These answers are of course "impossible" only by virtue of the verbal information provided. Furthermore, once the prediction table was set up, students could try to describe what features were invariant among the (valid) answers. In a sense, the data table allowed students to make a generalization.

The mere fact that they entertained an expression such as "N+3" as a reasonable way to express Mary's candies as a function of John's was encouraging. We believe students accepted this because the notation was a natural extension to the students' own experience. One student had introduced question marks to represent John and Mary's amounts. This led the teacher-researcher to point out the potential confusion due to using the same symbol, "?", to represent different amounts. This puzzle was advanced when students suggested that Mary's amount should be "any number plus three", which the teacher embodied first as "? +3" and later as "N+3".

Another teacher-researcher outright proposed that students adopt the letter N to specify an unknown amount. Once the students accepted this convention they came to the conclusion, through suggestions by two of the students, that Mary's amount should be called "N plus three." This is admittedly a small step in the direction of using algebraic notation; but we consider it an important one.

The second day of the candies activity took the discussion further into the territory of logical possibilities and constraints. In this class we asked the students to consider the totals of John's and Mary's candies, given the "fact" that Mary had three more candies than John. Could they have 15 candies, together? What about 12 candies? Or 2 candies? Why or why not? We gave them a number of problems to figure out based on varying assumptions about the totals. After working through several of these examples (with odd numbered totals), they began to see that there was a pattern: they could determine the number of candies in each box systematically. However, the fact that students systematically solved the problem (subtracting 3 and dividing the result by 2) did not guarantee that they could state their procedures clearly: at least two pairs of students in one class described their approach as dividing by 2 *and then* subtracting 3. Eventually, we would like to have seen them produce answers such as (N-3)/2, but they would need considerably more experience with using letters to stand for unknown amounts before such an expression would make sense to them.

AN EARLY APPROACH TO FUNCTIONS: THE MEDIATING ROLE OF COMPUTER-BASED ALGEBRA-LIKE SIGN SYSTEM

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Conceiving that the simultaneous variation of two variables in a numeric function table is governed by a general rule is an idea that many secondary pupils find very difficult. Even more difficult they find the idea that such variation can be analyzed and "manipulated" through expressing the corresponding rule in a symbolic (algebraic) way. The latter presupposes an appropriate understanding of numeric substitution in a formula as well as a basic notion of numeric variation domains. This means that from the perspective of the symbolic algebra, the mathematical sign system of algebra plays a crucial role as a mediator of subject's action, to build up the concept of functional variation. In some previous studies, this role of the notation system has been seen more as a constraint for pupils' thinking, rather than as a thought mediator.

However, in other studies, there has been gathered evidence with regards how algebra-like sign systems like spreadsheets can help 10-15 year olds to express and manipulate a functional relationship (Rojano & Sutherland, 1994). Results of this sort suggest that when working with a spreadsheet, children can resort to their numeric knowledge at the same time that they have to deal with a language that captures the generality of variation and functional dependence. This can be interpreted in the sense that the spreadsheet language takes on a mediating role in the pupils' development of an algebraic notion of function.

Cases from the spreadsheets study mentioned above corresponding to the youngest pupils (10 year olds) can be used as examples of how an early introduction to algebraic thinking can take place through a numeric grounded approach to functions. Whereas, issues from the same study with "algebra resistant pupils" (of 15 years of age) suggest that a connection between spreadsheets language and the algebraic code is feasible. In this way, a possible rout to algebra can be conceived, in which algebraic symbolism can be recovered as the language of generality and variation in school mathematics. Issues arising from a couple of cases will be used in this presentation to illustrate