concerning the study of the dynamic explorations and the transition between the conjecturing phase and the proof construction.

We think that the study on these processes will offer teachers tools that could be efficient to set up educational activities with the Mathematical Machines that can be productive for learning some geometrical topic but also for developing students' explorative and argumentative aptitudes.

References

Arzarello, F., Olivero, F., Paola, & D., Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments, *ZDM*, 34 (3), 66-72.

Bartolini Bussi, M. G. (2000). Ancient Instruments in the Mathematics Classroom. In Fauvel J., & van Maanen J. (eds), *History in Mathematics Education: The ICMI Study*, Dordrecht: Kluwer Ac. Publishers, 343-351.

Bartolini Bussi, M. G. (2005). The Meaning of Conics: historical and didactical dimension. C. Hoyles, J. kilpatrick & O. Skovsmose, *Meaning in Mathematics Education* Dordrecht: Kluwer Academic Publishers, 39-60.

Bartolini Bussi, M. G., Mariotti, M. A., & Ferri, F. (2005). Semiotic mediation in the primary school: Dürer's glass. in Lenhard J.; Seger F (eds). *Activity and Sign, Grounding Mathematics Education* (Festschrift for Michael Otte), New York: Springer, 77-90.

Bartolini Bussi, M. G., & Maschietto, M. (2008). Machines as tools in teacher education. In Wood, B. Jaworski, K. Krainer, P. Sullivan, & D. Tirosh (eds.), *International Handbook of Mathematics Teacher Education*, vol. 2 (Tools and Processes in Mathematics Teacher Education), Rotterdam: SensePublisher.

Bartolini Bussi, M. G., & Pergola, M. (1996). History in the Mathematics Classroom: Linkages and Kinematic Geometry. In Jahnke H. N, Knoche N., & Otte M. (eds), *Geschichte der Mathematik in der Lehre* , 36-67.

Bèguin, P., & Rabardel, P. (2000). Designing for instrument-mediated activity. *Scandinavian Journal of Information Systems*, 12, 173-190.

Boero, P. (eds.). (2007). *Theorems in school: From History, Epistemology and Cognition to Classroom Practice*, Sense Publisher.

Martignone, F., & Antonini, S. (to appear). Students' utilization schemes of pantographs for geometrical transformations: a first classification. *Cerme6*, Lyon, 2009.

Maschietto, M., & Martignone, F. (2008). Activities with the Mathematical Machines: Pantographs and curve drawers. *Proceedings of the 5th European Summer University On The History And Epistemology In Mathematics Education*, Univerzita Karlova, Prague.

Rabardel, P. (1995). *Les hommes et les technologies - Approche cognitive des instruments contemporains*. A. Colin, Paris.

MODELING AND PROOF IN HIGH SCHOOL

Mara V. Martinez & Bárbara M. Brizuela
University of Illinois at Chicago Tufts University

In this paper we propose a refinement of Chevallard's mathematical modeling process while still fully agreeing with his position that modeling is a key process in knowing mathematics. In doing this, based on our empirical study, we claim, 1) that there are at least two other stages in the mathematical modeling process; 2) that mathematical modeling is a non-linear process; and 3) there exists non-linearity at the interior of each modeling stage. We will illustrate each of these claims with episodes from our classroom intervention; one of the goals was to provide high school students the opportunity to use algebra as a modeling tool to prove.

INTRODUCTION

We fully agree with placing modeling at the center of knowing mathematics, following Chevallard (1985, 1989). While adopting this position, the goal of our paper is to work towards a refinement of Chevallard's mathematical modeling process. Chevallard (1989) proposed three stages in this process: (1) "*Identification of Variables and Parameters*;" (2) "*Establishing Relationships Among Variables and Parameters*;" and (3) "*Working the Model to Establish New Relationships*." While Chevallard provides the framework for our work, we will put forth three claims regarding a revised process: (1) there are at least two other stages in the mathematical modeling process: "*Interpretation of the Problem*" and "*Production of Competing Hypotheses*"; (2) mathematical modeling is a non-linear process; (3) the existence of non-linearity at the interior of each stage is evidenced by the presence of partial models. We will provide evidence for our proposed revision through an in-depth analysis of a triad of $9^{th}/10^{th}$ grade students (14-15 years of age approximately) who worked with the first author of this paper on a didactical sequence (the "Calendar Sequence") focused on students' use of algebra as a modeling tool to prove during fifteen one-hour-long lessons (see Martinez, 2008, in preparation; Martinez and Brizuela, under review).

DEFINITIONS OF MATHEMATICAL MODELING

Chevallard's definition of mathematical modeling

Chevallard (1989) described the three stages of the modeling process in the following way:

(1) We define the system that we want to study by identifying the *pertinent* aspects in relation to the study of the system that we want to carry out, in other words, the set of *variables* through which we decide to cut off from reality the domain to be studied … (2) Now we build a model by establishing a certain number of relations R, R', R'', etc., among the variables chosen in the first stage, the model of the systems to study is the *set*

In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International* for the Psychology of Mathematics Education, Vol. 4, pp. 113-120. Thessaloniki, Greece: PME.

of these relations. (3) We 'work' the model obtained through stages 1-2, with the go[al of] producing *knowledge* of the studied system, knowledge that manifests itself by m[…] relations among the variables of the system. (p. 53. Emphasis in original)

In Chevallard's (1985, 1989) point of view, mathematical modeling plays a key ro[le] in knowing mathematics. His theoretical perspective has been valuable [to] conceptualize the stages underlying mathematical modeling, as described above. Another important aspect is his consideration of both extra-mathematical and intra-mathematical contexts resulting in a broadening of mathematical modeling fo[r] problem situations both *inside mathematics* (intra-mathematical context) as well as in contrast to *real world contexts* (extra-mathematical context). In addition, of special interest to us is Chevallard's understanding of algebra as a modeling tool.

Additional perspectives on mathematical modeling

Lesh and colleagues have largely studied modeling with an emphasis on "real world" contexts (e.g., Lesh, et al., 2003; Lesh & Doerr, 2003; Lesh, Lester, & Hjalmarson, 2003; Lesh & Zawojewski, 2007). Lesh and his colleagues understand the development of models as part of problem solving. Students begin their learning experience by developing conceptual systems (i.e., models) for making sense of real-life situations where it is necessary to create, revise, or adapt a mathematical way of thinking (i.e., a mathematical model; Lesh & Zawojewski, 2007). Given model-eliciting activities, students are expected to bring their own personal meaning to bear on a problem, and to test and revise their interpretation over a series of modeling cycles. Lesh describes the modeling process as consisting of four interacting processes that do not occur in any fixed order: 1) *description*: establishing a map from the model world to the real or imagined world; 2) *manipulation* of the model to generate predictions or actions related to the problem solving situation; 3) *prediction* (or translation), that involves carrying back results into the real or imagined world; and 4) *verification* of the usefulness of actions or predictions.

Hanna and Jahnke (2007) also investigate modeling within extra-mathematical contexts. They use arguments from physics as a method to build an explanatory proof. According to Hanna and Jahnke (2007), "modeling often has to do with creating a non-physical representation of a physical system" (p. 147) and relate their approach to "reality related proofs." However, of special importance for our study is that they view modeling and proof as being inextricably linked and as having complementary roles. Additionally, they relate their view of modeling to that held by applied scientists, as, "a circular or spiral process of setting up a model, drawing conclusions, modifying the model, drawing conclusions, and so on" (p. 150). We borrow from Lesh and his colleagues their view of modeling as consisting of interacting stages that do not occur in a particular order, and from Hanna and Jahnke their view regarding the connections between modeling and proof, and their view of modeling as being a circular or spiral process.

[On] the other hand, other researchers (e.g., Bolea, Bosch, & Gascón, 1999; [Che]vallard, 1985, 1989; Gascón, 1993-1994), working more closely to Chevallard's [the]oretical perspective, place a central focus on the study of modeling in and of itself [as] well as on the meaning and implications of conceiving algebra as a modeling tool [in] the school curriculum. Within this group of authors, Chevallard, Bosch, and Gascón (1997) claim that an essential characteristic of a mathematical activity consists of building a (mathematical) model about systems (within intra-mathematical or extra-mathematical contexts) to be studied, to use it, and to produce an interpretation of the obtained results. In others words, the mathematical activity can be characterized as making (mathematical) models of systems in (intra or extra-mathematical) contexts. The authors underline three aspects involved in building a mathematical model: the routine utilization of pre-existing mathematical models, the learning of models as well as the way of using them, and the creation of mathematical knowledge. From this group of colleagues we borrow their vision of the importance of modeling not just extra-mathematical contexts but also intra-mathematical contexts, as well as their vision regarding algebra as a modeling tool central to the mathematics curriculum.

METHODOLOGY

Participants

In this paper, we will describe and analyze the work of three 9th/10th grade students (Abbie, Desiree, and Grace) who participated in a teaching intervention, led by the first author of this paper, in which a total of nine 9th/10th graders took part, at a public charter school in the Boston area, Massachusetts, in the USA.

Procedure

Lessons: Fifteen one-hour lessons were held once a week. These lessons were part of the regular school schedule but not part of their regular mathematics classes. In this paper we will report on data collected during Lessons 1 and 2, in which students focused on Problem 1, which was implemented during the first half of the intervention during which students worked with variables, algebraic expressions, and equivalent algebraic expressions.

Problem 1

Part 1: Consider a square of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be

1	2
8	9

These squares will be called 2x2 calendar squares. Calculate the difference between the products of the numbers in the extremes of the diagonals. Find the 2x2 calendar square that gives the biggest outcome. You may use any month of any year that you want.

Part 2: Show and explain why the outcome is going to be -7 always.

Figure 1. Problem 1 from the Calendar Sequence.

In Problem 1 (Figure 1), students were asked to analyze the nature of the outcome of the described calculation (subtraction of the cross product). It was expected that students would anticipate some kind of variation in the outcome in relation to the set of days where the operator is applied and that students would find out, through exploration, that the same outcome is always obtained, no matter where they apply the operator. The challenge for the students was to find out why this happens, and whether this is "always" going to be the case. At this stage in the problem, from a mathematical point of view, algebra becomes a tool to solve the problem. Thus, one of the challenges in Problem 1 is to show the limitations of using a non-exhaustive finite set of examples to prove that proposition is true, and to encourage students to use algebra as *a tool* that allows them to express all cases using a unique expression.

FINDINGS

Claim 1: There are at least two other stages in the mathematical modeling process.

Interpretation of the Problem: Given that our interest in mathematical modeling, specifically the use of algebra as a modeling tool to prove, is educational, it is crucial to include a stage that accounts for students' processes when they are mainly focused on understanding the statement of the problem, which does not exclude questions regarding the statement of the problem from re-appearing once students start to "solve" the problem. In Abbie, Desiree, and Grace's group, the first five minutes of small group work were characterized by trying to understand the meaning of the statement of the problem. Among the issues that surfaced during this interpretation stage were: potential strategies to solve the problem; reaching an agreement regarding the operations involved; the nature of the numbers considered (negative, positive, absolute value); and the delimitation of the domain to study. After these initial minutes, the students reached an agreement regarding these issues and their main focus shifted towards the production of hypotheses regarding the outcome.

Production of Competing Hypotheses: This stage is characterized by students' production of competing hypotheses as part of their investigation of the nature of the outcome. Students pondered how to obtain the largest number and how to characterize the nature of the dependence between the outcome and different aspects of the problem situations such as a square's position within the month, across months and years, and what day of the week is the first day of the month. In Figure 2, we show the process by which our three students produced a variety of competing hypotheses and later agreed on a final hypothesis. Students employed a significant amount of time trying to understand the behaviour of the outcome and its dependence/independence to different elements of the context. In the example given, students constructed a variety of competing hypotheses (H1, H2, H3, and H4 in Figure 2) until they were relatively convinced about the truth-value of their final hypothesis (H5). *What* has to be proved is not obtained through a straightforward

process, but rather through students' analysis of the likelihood of the truth-value of each of a succession of competing hypotheses. H1 linked the numeric outcome to the square's position at the beginning of the month, while H2 is a claim about the outcome's independence of the month that it was placed in. After abandoning H1 and H2, H3 was momentarily embraced: possibility of the outcome's dependence on the month where it is placed, which is in contradiction with H2. H4 seems to be a refinement of H3, given that it relates the potential variation of the outcome with what day of the week is the first day of the month. Students identify as the potential source for the variation in the outcome not only in what month the square is placed but also what day of the week is the first day in that month. The students hypothesize that a potential source of variation in the outcome is the relative arrangement of the days. To test their hypothesis, they tried months that differ in what day of the week is the first day of the month, always resulting in the same outcome, -7. Students seemed convinced given that they tried all possible unfavourable scenarios and they still got the same outcome, therefore they produced their concluding hypothesis (H5) stating the outcome is always -7 regardless of different arrangements of numbers, months, and first days of the month. We distinguish between two kinds of hypotheses: *exploratory* and *concluding*, the latter being the hypothesis to be proved. They differ in terms of the purpose and degree of certainty regarding their truth-value. An example of an *exploratory* hypothesis is provided by Desiree when she posed the following question to her group: "What happens if you start like at the beginning?" referring to the impact on the numeric outcome when placing the square at the beginning of the month. This is different from their concluding hypothesis as stated by Abbie: "They are always going to be the same." As discussed earlier, this concluding hypothesis was produced as the result of exploring various hypotheses, analyzing them using examples and properties of the relations involved, and getting a sense of the degree of certainty about its truth value.

Claim 2: Mathematical modeling is a non-linear process.

We claim that the mathematical modeling process is non-linear: stages can occur in *non-consecutive order*, and therefore does not necessarily follow the path as described by Chevallard (1985, 1989). As illustrated in Table 1, students already started analyzing some relations among variables (i.e., "*Establishing Relationships Among Variables*" stage) in the *Interpretation of the Problem* stage. In our analysis, it is possible to identify a main focus to students' work during a certain period of time. For instance, during the first five minutes, as the students were trying to understand the statement of the problem, elements from other modeling stages appeared. During "*Interpretation of the Problem*," Abbie noticed the relationships between elements in the corner of the diagonal of the square (i.e., "*Establishing Relationships Among Variables*"). At that moment, her group did not elaborate on her comment until later when they were mainly producing hypotheses; more specifically, until the moment of writing the expression to prove their concluding hypothesis, illustrating that stages do not necessarily happen in a fixed order.

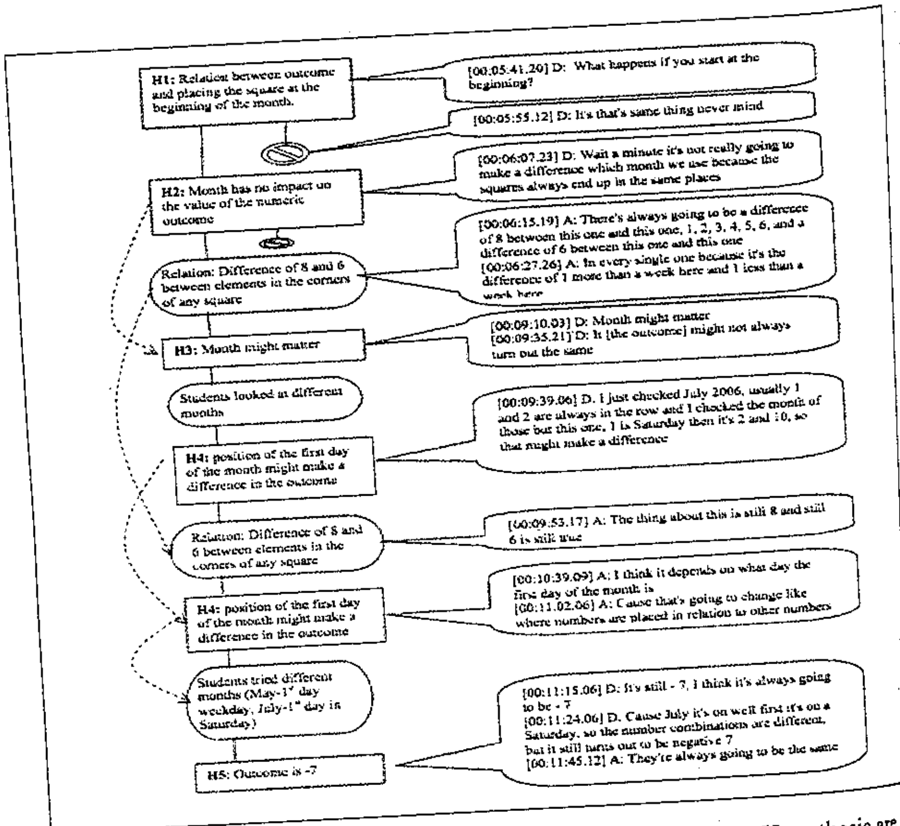

Figure 2. Proposed Production of Competing Hypotheses Stage. Hypothesis are represented on the left in order of appearance. On the right, excerpts of students' discussion along with the time code and the student initial.

Claim 3: Partial models within stages of the mathematical modeling process.

Even though Chevallard's mathematical modeling process has proved useful, it does not account for the complexity involved at the interior of each of the modeling stages. Within Chevallard's stages of *Identification of Variables and Parameters* (i.e., first stage) and *Establishing Relationships Among Variables* and Parameters (i.e., third stage), we have found that students produce *partial models* of the situation under consideration. A *partial model* is a model that includes some (but not all) variables, parameters, or relations among them. Thus, a *complete model* is what we would traditionally call model, and it includes *all* variables, parameters, and relations that are relevant to study the situation under consideration. For instance, building on the

relations identified when they first encountered the problem, Abbie proposed to write with "algebra" the following: "All right so this is what I have 'x' times 'x' minus 8,

Transcript	Stage	Significance
[00:05:41.20] and [00:06:07.23] Production of H1 and H2.	Production of Competing Hypotheses	New proposed stage.
[00:06:15.19] A: There's always going to be a difference of 8 between this one and this one, 1, 2, 3, 4, 5, 6, and a difference of 6 between this one and this one.	Establishing Relationships Among Variables	Chevallard proposed this as the final stage in the modeling process.
[00:09:10.03] Production of H3.	Production of Competing Hypotheses	New proposed stage.
[00:09:53.17] A: The thing about this is still 8 and still 6 is still true.	Establishing Relationships Among Variables	Chevallard proposed this as the final stage in the modeling process.
[00:09:39.06] Production of H4.	Production of Competing Hypotheses	New proposed stage.

Table 1. Illustration of non-linearity of the modeling process.

minus 'y' times 'y' minus 6 that's basically what we're doing, yes." At this moment they had produced a partial model of the situation since they were not aware of the relation between x and y. After producing that expression, they continued to check whether it was correct or incorrect; in order to do that, they were going to replace the letters with numbers. In the process of doing so, Abbie realized that y and x are in fact related: "'y' and 'x' aren't like random numbers … so we can say everything in terms of this number right here." This is when a complete mathematical model was produced, including one independent variable x and three dependent variables $x-1$, $x-8$, and $(x-1)-6$ with explicit relationships among the different variables. In this case, students' goal was to write down the expression representing the model that they thought captured all necessary information, in order to work on it with the ultimate goal of proving their concluding hypothesis. However, they came to realize that there were two relationships that they had overlooked. Therefore they "went back" to re-write these "unnoticed" relationships.

CONCLUDING REMARKS

In this paper, we have proposed a refinement of Chevallard's mathematical modeling process. We proposed the inclusion of at least two others stages namely: "Interpretation of the Problem" and "Production of Competing Hypotheses". In addition, we provided evidence illustrating the non-linearity (stages do not happen in a fixed order) of the mathematical modeling process. Lastly, we showed students' partial models as an indication of the complexity at the interior of the stages in the modeling process.

References

Bolea, P., Bosch, M., & Gascón, J. (1999). *The role of algebraization in the study of a mathematical organization.* Paper presented at the European Research in Mathematics Education, Osnabrueck.

Chevallard, Y. (1985). Le Passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au collége. *Petit X, 5*, 51-93.

Chevallard, Y. (1989). Le Passage de l'arithmétique a l'algébrique dans l'enseignement des mathématiques au collége. *Petit X, 19*, 43-72.

Gascón, J. (1993-1994). Un nouveau modele de l'algebre elementaire comme alternative a l'arithmetique generalisee. *Petit X, 37*, 43-63.

Hanna, G., & Jahnke, H. N. (2007). Proving and modeling. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education. The 14th ICMI Study* (pp. 145-152). NY, NY: Springer.

Lesh, R., Cramer, K., Doerr, H. M., Post, T., & Zawojewski, J. S. (2003). Mode development sequences. In R. Lesh & H. M. Doerr (Eds.), *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching* (pp. 35-58). Mahwah, NJ: Lawrence Erlbaum and Associates.

Lesh, R., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond Constructivism. Models and Modeling Perspectives on Mathematics Proble Solving, Learning, and Teaching* (pp. 3-33). Mahwah, NJ: Lawrence Erlbaum a Associates.

Lesh, R., Lester, Jr., F. K., & Hjalmarson, M. (2003). A models and modeling perspec on metacognitive functioning in everyday situations where problem solvers deve mathematical constructs. In R. Lesh & H. M. Doerr (Eds.), *Beyond Constructiv Models and Modeling Perspectives on Mathematics Problem Solving, Learning, Teaching* (pp. 383-403). Mahwah, NJ: Lawrence Erlbaum and Associates.

Lesh, R., & Zawojewski, J. (2007). Problem solving and modelling. In F. Lester (E *Handbook of Research in Mathematics Education* (vol. 2, pp. 763-804). Greenwich Information Age Publishing.

Martinez, M. (2008). Integrating Algebra and Proof in High School: The Case o Calendar Sequence. Paper presented at the 11th International Congress on Mathem Education, July 6-13, in Monterrey, Mexico.

Martinez, M. (in preparation). Integrating Algebra and Proof in High School: St Work with Multiple Variables and a Single Parameter.

Martinez, M. & Brizuela, B. M. (under review). Integrating Algebra and Proof School: Students' Work with Multiple Variables on The Calendar Sequence.

STUDYING TEACHERS' PEDAGOGICAL ARGUMENTATION

Metaxas Nikolaos, Potari Despina, Zachariades Theodossios

Department of Mathematics, University of Athens, Greece

We present a case study analysis of the arguments used by an experienced high-school teacher. We employ the model of argumentation schemes in adjunction with Toulmin's scheme. We focus on the content and the structure of the arguments used, and examine the different aspects of teacher knowledge that emerge. There is evidence of a structured example space which instantiates through the use of pedagogical examples as a conclusive and integral part of teacher's argumentation. This example space is based on a sense of deep pedagogical intuition framed by teacher's craft knowledge.

INTRODUCTION

There is a growing amount of research in mathematics education literature focusing on mathematical arguments (structures of inference with mathematical meaning) produced by students and teachers of mathematics. The main methodological tools for the analysis of these arguments are primarily Toulmin's model (Toulmin, 2003). Some researchers use Toulmin's model as a lens through which they document students' learning progresses in a classroom (Krummheuer 1995), while other researchers study the quality of a certain mathematical argument (Pedemonte, 2007). Nevertheless not much has been done in the direction of analysing the pedagogical arguments of teachers. In this paper we use Toulmin's model coupled with argumentation scheme analysis (Walton and Reed, 2005; Walton, Reed & Macagno, 2008) to dissect the structure of argumentation of mathematics teachers when they terpret and comment on students' answers and design teaching interventions to help dents overcome emerged difficulties. By examining the kind of arguments they ploy and studying their structure we seek to investigate some aspects of teacher wledge (Schulman, 1987; Ball, Thames & Phelps, 2007) that is lying beneath h argumentative scheme. So we examine the following questions: (a) what is the d of arguments teachers use, (b) what is the structure of these arguments and (c) t can be inferred about teachers' knowledge.

EORETICAL PERSPECTIVE

in's (2003) model asserts that most arguments consist of six basic parts, each ch plays a different role in an argument. The claim (C) is the position or claim argued for. The data (D) are the foundation or supporting evidence on which ument is based. The warrant (W) is the principle, provision or chain of ng that connects the data to the claim. Warrants operate at a higher level of ity than a claim or reason, and they are not normally explicit. The Backing (B)

ki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.). *Proceedings of the 33rd Conference of the International Psychology of Mathematics Education*, Vol. 4, pp. 121-128. Thessaloniki, Greece: PME.