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RELATIONSHIPS BETWEEN CHILDREN'S EXTERNAL REPRESENTATIONS OF NUMBER

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Previous studies of children's use of notation have pointed to different types of notational strategies (Alvarado, 2002; Brizuela, 2004; Cayton, 2007; Scheuer et. al., 2000; Seron & Fayol, 1994). In a recent study (Cayton & Brizuela, 2007) we found that first grade children were producing a great number of unconventional responses when writing large numbers. This study follows those same children into grade two to see how the children perform after another year of experience with writing larger numbers. We also examine the relationship between the children's written numbers and another type of external representation through valued tokens.

RATIONALE AND PAST RESEARCH

Studies of children's numerical understanding over the last decade suggest that there are identifiable progressions in how children develop number concepts (Cobb, 1997; Fuson, 1997). According to Fuson (1997), children construct meanings for numbers through the various interactions that they have with these numbers both in and out of school. Elementary school mathematics classrooms encourage or facilitate the development of various number concepts through the language that is used by the teachers and students, the type of materials that are used, the problems that are solved, and the class activities. These components act in concert with one another to support children's construction of meanings for numbers. One important and interesting aspect of numbers is that they can be represented in many different formats: written numerals, oral numbers, arrays of dots, tallies and more. What are the relationships, for children, among these representations of number?

In relation to written numbers, for instance, Bialystok and Codd (2000) ask, "what do children believe that written representations of quantity mean?" (p. 117). To illustrate the interconnections between written representations of number and other external number representations, studies have found that transcoding zeros within numbers proves particularly problematic amongst young children. For example, several recent studies (e.g., Cayton, 2007; Cayton & Brizuela, 2007; Scheuer et. al., 2000; Seron & Fayol, 1994) highlight that children's errors point to problems in both implementing the representational actions as well as in appropriating the number system itself.

In the study presented here, we wished to follow up to two previous studies. In Cayton and Brizuela (2007), presented at PME 31, we found that at the end of first grade, students were still producing a great number of unconventional responses in three different systems of external representations of number: oral, written, and a third system where numbers are represented by tokens of various colors, each a power of ten. In Cayton (2007), we found that two seemingly-similar strategies for written numbers (Full Literal Transcoding [FLT] and Compacted Notation [CN], described below) produced by kindergarten and first grade children were, in fact, associated with different strategies in token-building: children using CN tended to perform similarly to children writing conventional numbers, with a large number of children building conventional token arrangements; while children using FLT were more likely to form unconventional token arrangements. We wondered if we would find this same pattern with older children, who have received more instruction and have more experience with written numbers and larger numbers in general.

METHOD

Participants

Twenty-six second grade students (students need to be seven years of age by the time they begin second grade) were interviewed individually. The school is located in an urban suburb of the United Stated of America. The school is ethnically, racially, and socio-economically diverse. In addition, the school provides a two-way bilingual education to children. All children in the second grade were invited to participate. Only children whose parents consented to their participation were included in the study.

Materials and Procedures

Interviews were carried out as clinical interviews (Piaget, 1965). During the course of the interviews, children were presented with the numbers detailed in Table 1. Our goal was to be able to explore children's oral, written, and nonverbal representations of number. Our proposal was to access children's oral representation through their oral naming of numbers; their written representation through their writing of numbers; and their nonverbal representations through their construction, through tokens, of the "value" of the different numbers.

Each of the interviews has three tasks: oral, written, and tokens. Each one of these tasks has both a production and interpretation mode: when numbers are presented by the interviewer in tokens, they can be interpreted through writing or through naming orally; when numbers are presented by the interviewer in writing, they can be interpreted through construction of tokens or through naming orally; when numbers are presented by the interviewer orally, they can be interpreted through construction of tokens or through writing.

Oral task: In this part of the task, children were asked to read from a piece of paper or from a token composition the numbers in Table 1.

Written task: In this part of the task, every child was asked to write at least two numbers from each series in Table 1 after being presented the number orally or through tokens.

Tokens task: This part of the task was designed for the purpose of understanding the consistencies/inconsistencies in the child's understanding of our number system without the use of notation. Children were presented with tokens of different colors.

Tokens were chosen based on the work of Nunes Carraher (1985) performing similar tasks in the understanding of place value in young children and illiterate adults. The child was told that red tokens are worth 1 point, blue tokens are worth 10 points, white tokens are worth 100 points, brown tokens are worth 1,000 points, and maroon tokens are worth 10,000 points. The child was asked to compose a number from Table 1 with the tokens after being presented the number orally or in writing.

Series	Number Type		
Series 1	Three digit – without 0	127	143
Series 2	Three digit – internal 0	101	207
Series 3	Three digit -- final 0	300	760
Series 4	Four digit – without 0	1127	3143
Series 5	Four digit – X0XX	3064	2053
Series 6	Four digit – XX0X	2101	3504
Series 7	Four digit – XXX0	1300	3760
Series 8	Five digit–without 0	21127	13143
Series 9	Five digit--XX0XX	43064	52053
Series 10	Five digit—XXX0X	22101	33504

Table 1. Numbers presented to children in the three different tasks (orally, through tokens, or in writing). Numbers were designed based on the work of Alvarado and Ferreiro (2002), Power and Dal Martello (1990), and Seron and Fayol (1994)

Children were randomly assigned to one of six task orders: (a) two possible conditions with an *oral introduction*; (b) two possible conditions with a *written introduction*; (c) two possible conditions with a *token introduction*. See Cayton and Brizuela (2007) for more details on the tasks and conditions.

ANALYSIS

All the interviews were videotaped. Transcripts of the interviews were reviewed along with any notes made during or after the interview, the written work of the children, and the physical manipulations of the children during the tokens tasks as documented in the videos. These pieces constituted the data for the study. This paper only looks at the analysis of two of the study tasks: written and token.

Data was arranged into categories for different types of strategies. In the written task, responses were classified by the strategy used to produce each written numeral. There were eight categories coded:

A) *Idiosyncratic* - No discernable strategy used for writing the number. For example, in one instance, the number 153 was transcribed as 4033.

B) *Missing Digits* - The written numeral is missing digits from the original number, either replaced by zero or deleted entirely. For example, the number 1127 could be represented as 1027 or 127.

C) *Digit Transposition* - Two or more digits of the number are transposed with one another. All of the original digits are still contained in the number. For example, the number 1127 could be represented as 1217.

D) *Full Literal Transcoding (FLT)* – Child writes out number literally, for example, 100701 or 10071 for one hundred seventy-one. This category is taken from Seron and Fayol (1994). This is similar to the Scheuer et al (2000) category of logogramic notation except that Scheuer would only allow for 100701 to be considered in this category. I allow for both types of literal transcoding (100701 and 10071) as FLT since for children who are conventionally writing 2-digit numbers, "71" has become the literal writing of seventy-one.

E) *Compacted Notation (CN)* - Child writes extra zeros in numbers but fewer than the FLT notation, for example, 1071 for one hundred and seventy-one. This category is taken from Scheuer et al (2000).

F) *Error due to incorrect use of comma* - This category only pertains to numbers over 999. The child uses a comma in notation and leaves out zeros. For example, one thousand seventy-one would be written 1,71.

G) *Lexical Error* - Child replaces one digit of the number with a different digit. For example, 137 for 127.

H) *Conventional Response* - Number conventionally represented.

In the tokens task, children were classified by the strategy used to compose the point value of the tokens. Composition number strategies were coded separately for each type of number. The same eight categories were used for each type of number:

A) *No Response* - Child answers "I don't know" and will not provide a guess.

B) *Incorrect Understanding* - Child fails to understand the multiplicative nature of the tokens. While in previous studies with Kindergarten and first graders, the most common example of this was counting every token as one point regardless of its color; this was rarely seen with second graders. The most common example of incorrect understanding in second grade was lining up the tokens in a literal order. For example, 1300 would become one one-point token, one thousand-point token, three one-point tokens, and one-hundred-point token to make "one- thousand - three - hundred."

C) *Counting by ones* - Child does show some understanding of token value, but can only add up the points by counting by ones (i.e., pointing to a 10-point token and counting 1 to 10), always leading to an error when dealing with large numbers.

D) *Incorrect token value* - One value of token was replaced entirely with a token of a different value. For example, the number 127 could be composed with tokens totaling 1027.

E) *Value Missing* - One value of token is missing entirely, with the other values all having the correct total. For example, the number 1127 could be composed to total 1027 due to missing one hundred-value token.

F) *Non-canonical, incorrect total* - Responses where there were more than nine tokens of any single value were classified as non-canonical. In this category, responses were both non-canonical and the total value was incorrect. For example, in composing the number 127, the response could have twenty ten-point tokens and seven one-point tokens, totaling 207.

G) *Non-canonical, correct total* - Responses were non-canonical, but the total value of the tokens was correct. For example, in composing the number 127, the response could have twelve ten-point tokens and seven one-point tokens, totaling the stimulus value of 127.

H) *Canonical and correct* - Number was represented canonically (no more than 9 tokens of any given value) and with the correct total value.

RESULTS

A total of 494 written numbers and 508 token constructions were produced by the twenty-six children. While a majority of written numbers were conventionally written by the second graders (375 of 494, 75.9%), the number of conventionally written numbers dropped dramatically as the length of the number increased, from 145 of 150 (96.7%) of 3-digit numbers to 141 of 200 (70.5%) of 4-digit numbers, and finally 89 of 144 (61.8%) of 5-digit numbers (see Table 2).

While this may seem logical, given that children have more practice with smaller numbers, we could argue that once children have appropriated the rules of the number system, numbers of any length should be equally accessible. Further, the difference in percentage of responses correct between four- and five-digit numbers indicates that this is not due to misunderstanding the word "thousand" or other vocabulary as both of these use all of the same terminology.

Category of Response	Three-digit	Four-digit	Five-digit	**Total**
Idiosyncratic (Idio.)	2	7	2	**11**
Missing Digits (MD)	0	2	16	**18**
Digit Transposition (DT)	1	1	2	**4**
Full Literal Transcoding (FLT)	0	24	17	**41**
Compacted Notation (CN)	2	13	14	**29**
Error due to Comma (Com.)	N/A	2	2	**4**
Lexical Error (LE)	0	10	2	**12**
Conventional Response (Conv.)	145	141	89	**375**
Total	**150**	**200**	**144**	**494**

Table 2. Written number responses by children, n=494

The tokens task produced similar results, with only one three-digit number of 156 total responses not produced conventionally (in fact, the one response was a refusal, not an incorrect representation). With 4-digit numbers the conventional rate dropped to 179 of 207 (86.5%) and 121 of 145 (83.4%) 5-digit numbers (see Table 3).

Category of Response	Three-digit	Four-digit	Five-digit	**Total**
No Response	1	1	4	6
Incorrect Understanding	0	10	10	**20**
Counting by Ones	0	3	1	**4**
Incorrect Token Value	0	8	4	**12**
Value Missing	0	2	2	**4**
Non-canonical, Incorrect Total	0	3	3	**6**
Non-canonical, Correct Total	0	1	0	**1**
Conventional Response	155	179	121	**455**
Total	**156**	**207**	**145**	**508**

Table 3. Token compositions by children, n=508

We next compared the tokens results with the written number strategies to investigate whether specific strategies in each representational system were associated to one another. The entirety of our cross-tabular comparisons will be discussed at length in following papers, but for now, we focus our attention to the two notational strategies, FLT and CN, in which we found disparities in previous studies (see Table 4).

Table 4 shows FLT and CN responses in relationship to unconventional (all categories except for the "Conventional Response" category) and conventional token constructions. The results and differences were striking, while 92.6% of CN responses were associated to conventional token constructions, only 62.5% of FLT responses were associated with conventional token constructions [χ^2 (1, N = 67) = 7.71, p<.01].

	FLT	CN	**Total**
Unconventional	15 (37.5%)	2 (7.4%)	**17 (25.4%)**
Conventional	25 (62.5%)	25 (92.6%)	**50 (74.6%)**
Total	**40 (59.7%)**	**27 (40.3%)**	67 (100%)

Table 4: Unconventional and conventional token compositions amongst FLT and CN written number responses, n=67

DISCUSSION

While these results do replicate our previous findings with younger children in which we found that FLT and CN were associated with different strategies in token-building: children using CN tended to perform similarly to children writing conventional numbers, with a large number of children building conventional token arrangements; while children using FLT were more likely to form unconventional token arrangements; we still consider it to be quite puzzling: why is it that notations for numbers of the type CN tend to be related to more conventional number constructions (through tokens) than FLT, seeing that both of these strategies are incorrect, never taught, and both formed by the over-use of zero in numerical notation? In the case of FLT, one could call it the numerical equivalent of sounding out the spelling of a word, which children are taught to do in written language. If this is the case, it still does not explain the prevalence of CN. Why do children choose to eliminate some but not all zeros in a number? Moreover, why do children who are performing this way appear to be closer to a conventional understanding of base-ten number construction? Seron and Fayol (1994) posit that perhaps children are absorbing some rules of notation, such as the "overwriting" of zeros, but are not yet grasping the entire concept of the system.

This leads us to the next obvious question, which is: are numerical notation strategies shaped by understandings of number construction or does the notation influence what children understand about numbers (i.e. their understanding of the base-ten construction of numbers)? The finding that numbers represented by conventional token constructions were much more variable in notational strategies than the reverse seems to point in the direction of the notation as a reflection of the child's constructional understanding. That being said, we still have much to learn about how children understand the unique and global system of numerical notation. The data presented in this paper show how much we may be able to learn about how children understand various aspects of the number system by looking at their notational strategies.

The data here also indicate that by the end of the second grade, children are still having difficulty in representing numbers. This should be a great cause for concern as this is the same age in which children begin learning place-value algorithms for arithmetic, and yet, they are demonstrating an incomplete understanding of the use of place-value and of written numbers and what they represent.

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