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A third grader's way of thinking about linear function tables☆

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Abstract

This paper is inscribed within the research effort to produce evidence regarding primary school students' learning of algebra. Given the results obtained so far in the research community, we are convinced that young elementary school students can successfully learn algebra. Moreover, children this young can make use of different representational systems, including function tables, algebraic notation, and graphs in the Cartesian coordinate grid. In our research, we introduce algebra from a *functional* perspective. A functional perspective moves away from the mere symbolic manipulation of equations and focuses on relationships between variables. In investigating the processes of teaching and learning algebra at this age, we are interested in identifying meaningful teaching situations. Within each type of teaching situation, we focus on what kind of knowledge students produce, what are the main obstacles they find in their learning, as well as the intermediate states of knowledge between what they know and the target knowledge for the teaching situation. In this paper, we present a case study focusing on the approach adopted by a third grade student, Marisa, when she was producing the formula for a linear function while she was working with the information of a problem displayed in a function table containing pairs of inputs–outputs. We will frame the analysis and discussion on Marisa's approach in terms of the concept of theorem-in action (Vergnaud, 1982) and we will contrast it with the scalar and functional approaches introduced by Vergnaud (1988) in his Theory of Multiplicative Fields. The approach adopted by Marisa turns out to have both scalar and functional aspects to it, providing us with new ways of thinking of children's potential responses to functions.

Keywords: Function tables; Algebra; Elementary school students; Theorems-in-action

1. Introduction

Research in mathematics education (Bednarz, Kieran, & Lee, 1996; Dubinsky & Harel, 1992; Schliemann, Carraher, & Brizuela, 2007) has shown the importance of a functional perspective in the teaching of algebra. A functional approach to algebra stands in contrast to a perspective that focuses on the symbolic manipulation of equations, usually referred to as an equational approach (see Chazan & Yerushalmy, 2003). The concept of function can facilitate the introduction to algebra through the use of different ways of representing functions: algebraic notation, function tables, and graphs in the Cartesian coordinate grid. In addition, an early introduction to a functional perspective can foster a deep insight into

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the concept of function, central in the field of mathematics. From a functional perspective, functions include equations as only one element of the teaching and learning of algebra.

Furthermore, past research has also provided examples of elementary school students' emerging understanding of functional relations (e.g., Schliemann, Carraher, & Brizuela, 2001; Schliemann, Goodrow, & Lara-Roth, 2001; Schliemann et al., 2007), showing that young students are able to begin to think functionally and to make use of algebraic notation. In particular, function tables and graphs have been shown to encourage children to focus on functional relationships (e.g., Schliemann, Carraher, et al., 2001; Schliemann, Goodrow, et al., 2001; Schliemann et al., 2007).

In our research, we are interested in describing, specifying, and analyzing what kind of knowledge students can construct within a functional approach to algebra. In particular, we are interested in the scope of validity[1] of the knowledge *produced by students* and how it differs from the target knowledge of the lessons *presented to the students*. With this data we hope to tailor ways to create bridges between the students' knowledge and the target knowledge for the didactical situations we design. By target knowledge we mean what Vergnaud (1994) refers to as, "real theorem[s], as science would see it" (p. 225). Vergnaud (1996), in his Theory of Conceptual Fields, proposes the concept of *theorem-in-action*. This turns out to be a very useful concept in the kind of approach and analysis mentioned above, because it allows for the explicit differentiation and connection between student knowledge and target knowledge. Vergnaud (1996) provides the following definition for theorems-in-action:

> A *theorem-in-action* [italics in original] is a proposition that is held to be true by the individual subject for a certain range of the situation variables. It follows from this definition that *the scope of validity of a theorem-in-action can be different from the real theorem, as science would see it* [italics added]. It also follows that a theorem-in-action can be false. (p. 225)

In his analysis of the multiplicative conceptual field, Vergnaud differentiates between two approaches: the scalar and the functional. In this paper, we analyze Vergnaud's Theory of Conceptual Fields by interpreting Marisa's theorem-in-action, which is neither scalar nor functional, while at the same time adopting some aspects of both.

Past research has shown that approaches other than the scalar and functional are possible. In their research in Brazil with street sellers with relatively little schooling, Nunes, Schliemann, and Carraher (1993; see also Schliemann et al., 1998), for instance, found that the sellers they interviewed computed the price of a certain amount of items by performing successive additions of the price of one item, as many times as the number of items to be sold. Nemirovsky (1996) also provides an example of an approach that is not entirely scalar or functional in a seventh grader he interviewed.

Identifying children's spontaneous theorems-in-action allows us to design instructional situations that can bridge the distance between children's knowledge and the target knowledge. Vergnaud (1988) has described the importance of identifying students' theorems-in-action as a way of uncovering their intuitive knowledge, to make connections to explicit mathematical content, and to make recommendations for teaching. Theorems-in-action have been described as "mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem" (Vergnaud, 1988, p. 144). These theorems-in-action are usually not expressed verbally, most of them are not explicit, and they may even be wrong. Other researchers working within the Piagetian theoretical framework have referred to these theorems-in-action as hypotheses (Ferreiro, 1986) or theories (Karmiloff-Smith & Inhelder, 1974). Ferreiro (1986), for instance, has used the term "hypothesis" to refer broadly to ideas or systems of ideas constructed by children to explain the nature and way of functioning of written language as an object of knowledge.

In this paper, we present the case study of a third grade student, Marisa, learning algebra from a functional perspective and working with the information displayed in a function table. In this context, our research questions were: How do students work with linear function tables? What kinds of approaches do students adopt while working with linear function tables? Marisa used a theorem-in action that was unexpected for us. Her theorem-in-action seems to be hybrid, between the scalar and functional approaches described by Vergnaud (1988, 1996).[2] We will discuss the

[1] By *scope of validity* we mean the domain of application of particular knowledge within the domain of mathematics.

[2] In French, "fonction linéaire" is used only and exclusively for functions $f(x) = ax$. The expression "fonction affine" is used for functions $f(x) = ax + b$. In English, "linear function" covers both kinds of functions. In his work, Vergnaud focuses on "fonction linéaire" [$f(x) = ax$]; this implies that the case that we analyze in this paper, that deals with "fonction affine," was not specifically analyzed by Vergnaud.

features that Marisa's theorem-in-action shares with both approaches (scalar and functional), analyze its validity, and its relationship with the target knowledge intended for the lesson.

2. Methodology

2.1. General approach

Our work with Marisa is inscribed within the Early Algebra (EA) Project. Within the project, we try to promote the learning of algebra through problems in *extra-mathematical* contexts, as opposed to *intra-mathematical* contexts (see Chevallard, 1989). By *extra-mathematical* we mean contexts where a problematic situation is raised using phenomena external to the mathematical field. In order to be solved, this context or situation can be modeled using mathematical tools. In other words, the issues to be solved are not about the nature of the mathematical objects themselves, nor about the relations among these objects. The objects on which the relations operate have an intended meaning beyond mathematics. For example, in the set of problems that we focus on in this paper, the context is a restaurant where there are square dinner tables that are detached, and we are interested in counting the amount of people that can be seated in a given number of tables. The objects of the problem (maximum number of people that can be seated and number of tables) are not mathematical objects. They are objects or sets of objects that can be quantified, and it is in this sense that we are mathematizing the problem, and that we can begin to study the relations among these numbers that are representing a property (cardinality) of a set made up of non-mathematical objects (e.g., chairs, people, etc.). In contrast, *intra-mathematical* problems are those problems that appear purely from mathematical questions and problems. Chevallard (1989) introduced the concept of intra-mathematical modeling and Sadovsky (2005) explains this concept in the following way:

> Chevallard ... also revindicates the concept of modeling to think about the production of knowledge in one mathematical system through another mathematical system. He calls this 'intra-mathematical modeling'. (p. 27)

Nemirovsky's (1996) contrast between contextualized and decontextualized problems parallels the extra- and intra-mathematical distinction, reminding us of the fuzzy distinction between both in the following way:

> Often problems are characterized as being decontextualized because they are just about numbers (as opposed to quantities or measures of specific things), as if all the rich background of ideas and experiences that students develop around numbers could not offer a context. I see the origin of this mistaken notion in the assumption that the context belongs to the formulation of the problem, ignoring that real contexts are to be found in the experience of the problem solvers. (p. 313)

Intra-mathematical contexts are those in which only numbers and letters are presented, with no grounding in a specific everyday experience. However, as Nemirovsky (1996) points out, we must also bear in mind that in most likelihood, at least some of the students will ground the numbers and letters in everyday experiences, spontaneously, for themselves. That is, some students may provide *extra-mathematical* contexts for *intra-mathematical* problems.

Vergnaud (1982, 1988, 1994) suggests that intellectual complexity can be gained by learning to manage new types of situations, such as different contexts or types of problems. Nemirovsky (1996) also argues that new types of situations, in the form of real world problems, offer fruitful contexts for students' learning to deal with complexity. It is in response to this type of suggestion that these *extra-mathematical* contexts are included in systematic ways in our design of problems for our EA lessons. In general, most of the problems we develop and present can be modeled by a linear function.[3] These problems and contexts offer a setting in which to closely analyze and understand different ways of thinking that children exhibit. In addition, we try to find the coherence in the different approaches adopted by children ("to give the child reason," as Duckworth [1987, p. 87] put it), as well as in the systems of thinking that might have brought forth these approaches.

[3] During the 2004–2005 academic year (when the children described in this paper were in the fourth grade), we also introduced phenomena that could be modeled by a quadratic function. In addition, the students worked with phenomena presented through a Cartesian coordinate graph that did not correspond to the graph of a polynomial function.

2.2. Data collection and analysis

The data for this paper is drawn from a third grade mathematics classroom in an urban public school in the Boston, Massachusetts (United States of America) area. The classroom was composed of 15 students that the EA Project worked with during the 2003–2004 school year. As members of the EA Project, we went into this third grade classroom twice a week for 50 minutes each session. The EA classes these children participated in were in addition to their regular mathematics classes. In addition to these EA lessons, two EA homework sessions were held by the regular classroom teacher each week, reviewing the problems assigned by the EA project team members. The children in this study entered the EA project at the beginning of their third grade. They had a total of 21 EA lessons in the Fall semester, and 30 in the Spring semester (see www.earlyalgebra.terc.edu for a full description of these lessons). All classes were videotaped by two members of the research team, each handling a video camera. An additional team member taught each one of the lessons. Occasionally, a fourth team member would observe the lesson. In addition to videotapes, we collected students' written work for each one of the lessons. The format of the lessons offered moments of large group discussion as well as time for small group work, in which small group and individual on-the-spot interviews about the children's work were carried out. This paper focuses on three particular lessons that took place in the Spring 2004 semester of third grade, as well as on an individual interview that was carried out in June 2004 with one of the students in the class, Marisa. The data analysis is based on the analysis of lesson and interview tapes, and Marisa's written work. We chose these particular lessons because: all the lessons focused on problems that were variations on the same theme (dinner tables); all the problems had the same underlying mathematical structure of a linear function; all the lessons focused on the use of function tables; and the target problem provided a rich extra-mathematical context in which to explore students' approaches to functions and function tables.

3. The algebra lessons taught and children's approaches to functions during the lessons

3.1. Lesson 34[4]

In this lesson, the problem presented to the class dealt with a restaurant that only has square tables.[5] Each of the tables at the restaurant is arranged separately, and they cannot be put together. This is the statement of the problem as it was presented to the children:

> In your restaurant, a maximum of four people can sit at each dinner table.
>
> Fill in the following data table. (See Fig. 1.)
>
> If you know the number of tables, figure out the maximum number of people you can seat.
>
> If you already know the number of people, figure out the minimum number of tables you need.

In Fig. 1, the function table that the students had to complete is shown.

In this particular problem, the children were asked to figure out the maximum number of people that could be seated (dependent variable), in relation to the number of dinner tables (independent variable). The children were asked to do this by completing a function table (see Fig. 1), as well as by focusing on the question about x dinner tables. The function that describes the relationship in this case is

$$f : N_0 \rightarrow R, \quad f(x) = 4x,$$

where the domain of the function is the set of natural numbers including zero, and x stands for the number of dinner tables. The homework assigned for this lesson was a variation of lesson 34, in which the shape of the dinner tables was triangular instead of square.

[4] Lessons were numbered beginning at 1 in the first lesson taught in the Fall 2003 semester, in third grade.
[5] See Nemirovsky (1996) for a description of a seventh grader's response to this problem.

Number of Dinner Tables	Show How	Number of People
1 ⟶	1 × 4	
2 ⟶	2 × 4	
3 ⟶		
4 ⟶		
⟵		24
⟵		20
⟵		11

How many people can you sit at x tables?

Fig. 1. Function table presented to children in lesson 34.

3.2. Lesson 35

Lesson 35 was a follow-up of the previous one, and therefore very similar in nature. In this case, the square dinner tables could be put together, so the function describing the relationship between independent and dependent variables is not proportional, as it had been in the previous lesson, number 34:

$$f : N_0 \to R, \quad f(x) = 2x + 2,$$

where R is the set of real numbers and N_0 is the set of natural numbers with zero.

The homework given to the children after this lesson asked them to consider a case of a restaurant that only had hexagonal dinner tables that were arranged in a line, side by side. Each one of these lessons is a variation on a similar theme (i.e., the dinner tables) and on the underlying mathematical structure of a linear function. By exposing children to many problems with similar structures, our intent was to highlight a general rule that could help children describe the relationship between variables in this type of problem.

3.3. Lesson 51 (review)

This lesson took place at the end of the Spring 2004 semester. It was a review in preparation for an assessment the children were scheduled to take a few weeks later, as well as a way to compare the children's responses and performance on two similar tasks at two different moments in the semester (earlier [i.e., lesson 35] and later [i.e., lesson 51] in the Spring semester). This lesson was a variation on lesson 35. In lesson 35, larger numbers, like 100 and 200, and the case for x, a variable number of dinner tables, were not explicitly considered in the function table that children were asked to complete. In lesson 35, the project team member who had been teaching the lesson did bring up these cases in the large group discussion, even though it was not part of the planned activities and goals. In lesson 51, numbers such as 100 and 200, as well as x, were *explicitly* included in the function table presented to the third grade children.

For homework, the students were asked to complete a function table, where the independent variable was the number of dinner tables and the dependent variable was the maximum number of people that could be seated. The information was given through a Cartesian coordinate graph. The children were also asked to hypothesize about the shape of the dinner tables and whether or not they were attached. The function in this case was:

$$f : N_0 \to R, \quad f(x) = 3x.$$

3.4. Strategies used by students during lessons 35 and 51

Given the importance of identifying students' theorems-in-action as a way of understanding their intuitive knowledge, to make connections to the target knowledge for the lesson, and to make recommendations for teaching (Vergnaud, 1988), we sought to identify the theorems-in-action adopted by students as they approached the dinner tables problems described above. Within the range of theorems-in-action used by the children, we identified two that had previously been described in the literature (Vergnaud, 1982, 1988, 1994).

3.5. The scalar theorem-in-action

Vergnaud (1994) describes this theorem-in-action as being introduced through iterated addition[6]; it therefore relies on the additive isomorphism property from which the multiplicative isomorphism property is derived. In the context of the problem used in lessons 35 and 51, this theorem-in-action consists in adding by twos while going down the columns in the function table. In order to get $f(3)$, the child does $f(2) + 2$. In general terms, you can solve for

$$f(x) = f(x-1) + 2$$

by simply knowing $f(1)$.

This theorem-in-action is an easy and quick way of finding out the values of the dependent variable (e.g., the maximum number of people that can be seated at the dinner tables), because students just have to add two to the previous row in the function table. Note that, in order to use this theorem-in-action, a "first" value for the dependent variable is required. Using this theorem-in-action, children only have to apply the algorithm "adding two." *What* they are adding two to remains implicit and can be achieved by just using the values for the dependent variable and maintaining the independent variable implicit. With this theorem-in-action the child does not need to keep track or use any information about the independent variable—in this case, the number of dinner tables. The child that is using this kind of theorem-in-action is focusing on the fact that each time that one more dinner table is added, two more people can be seated; or, the child can be paying attention to the "adding two" operation in the dependent variable column while ignoring the independent variable.

This theorem-in-action is effective while we have a sequence of natural number values in the independent variable column; if a number is skipped in the function table, for instance from 6 dinner tables to 29 dinner tables, children cannot add 2 to the value associated with 6 to figure out the output value for 29. One possibility to overcome this shortcoming would be for the child to add multiples of two; in this case, for instance, adding two twenty three times, if the children are able to figure out that they need to "count-by-twos" 23 times, which is the difference between the two values they do have (i.e., $29 - 6 = 23$).

By using this scalar theorem-in-action, a child is not likely to come up with a general formula to get the maximum number of people knowing the number of dinner tables. Children who use this theorem-in-action do not seem to bear in mind the values for the independent variable when performing these calculations, and they do not even use these values to implicitly calculate the number of people. That is, they do not express the *relationship* between the two variables.

3.6. The functional theorem-in-action

Vergnaud (1994) explains that this theorem-in-action uses the constant coefficient property instead of the previous isomorphism property. Using the context and the geometrical arrangement of the dinner tables to infer the maximum number of people that can be seated at the square dinner tables, for instance, the students who use this theorem-in-action multiply the number of dinner tables by 2 and then, add 2. These children use a theorem-in-action that yields the maximum number of people that can be seated depending explicitly on a particular number of dinner tables. This theorem-in-action can be described as functional because children explicitly relate the numeric value of the independent variable (e.g., maximum number of people that can be seated) to the numeric value of the dependent variable (e.g., number of dinner tables). In most cases, the students in our EA class were not able to express the relationship without using a *particular* value; that is, they were not able to construct a general rule or formula that would apply to all cases (see Martinez, Schliemann, & Carraher, 2005). It is when using a functional theorem-in-action, versus a scalar theorem-in-action, that children are more likely to be able to generalize the relationship between the dependent and independent variables beyond particular cases, and think of cases involving large numbers and a variable quantity such as x.

[6] This is a particular case of recursion where $a_n = a_{n-1} + 2$ for all $n \in N$.

In the context of this dinner tables problem, a third type of theorem-in-action arose, reflecting another way of thinking about the problem. The literature provides some indications that other approaches to linear functions are possible. As mentioned earlier, Nunes et al. (1993; see also Schliemann et al., 1998) describe the strategies adopted by street sellers in Brazil, who compute the prices for different amounts of items by performing successive additions of the price of one item, as many times as the number of items to be sold. Nemirovsky (1996) also describes a student's (Damian) approach to the dinner tables problem. Damian, who had just completed seventh grade and had not yet taken algebra, used addition to compute the number of people who can sit at a certain number of dinner tables. Damian constructed a table, in which, "the column to the left indicated the number of unit-tables, whereas the column to the right is the number that needs to be added to the one on the left to obtain the amount of available places to sit" (p. 301). Damian noticed that the number to the right is always two more than the number of tables.

These previous descriptions provided by the literature are essential to us. Being aware of and understanding the variety of students' approaches to problems is helpful as well as necessary for us as educators and researchers. Vergnaud (1994) explains that,

> Theorems-in-action are a way for us to analyze students' intuitive strategies and to help them in transforming intuitive knowledge into explicit knowledge. They also provide a way for us to make a better diagnosis of what students know or do not know so that we may offer them situations that will enable them to consolidate their knowledge, increase it, recognize its limits, and eventually overtake it. (p. 149)

Thus, previous descriptions (in the literature, for instance) of children's theorems-in-action and strategies provide *expectations* for ways children could respond and react.

4. Marisa's theorem-in-action

The theorem-in-action we will describe focuses on Marisa's conceptualization of the relationships embedded in the dinner tables problem used in lessons 35 and 51. Marisa was a quiet third grade student that worked hard and was not labeled as a brilliant student at school; thus, we could think of her as a typical third grade student. Marisa usually held strong convictions about the problems she was working on and she was always very careful to justify these convictions. When presented with an alternative perspective or explanation for a problem, she would listen carefully but would not change her mind until she was absolutely sure she understood the change thoroughly and could explain it herself. Marisa's way of conceptualizing the relationship between maximum number of people seated and number of dinner tables is *neither scalar nor functional*. At the same time, it is *both* scalar and functional. As such, it can be thought of as a *hybrid* theorem-in-action. We use the word hybrid because Marisa's approach shares features with both the scalar and functional approaches. By hybrid we do not mean to imply that it is a transitional moment, between the scalar and functional. At this point, we do not have enough evidence to claim that there exists a progression through each of these three approaches (i.e., scalar-hybrid-functional). It is our intent to show that there are other potential approaches to functions, beyond those that we might expect.

During lesson 51, and while working with the function table shown below (see Fig. 2), Marisa said that she saw "another pattern," besides the scalar and functional relations that her peers were describing. The researcher who was teaching this particular lesson [the second author of this paper] asked her to explain the pattern she saw:

> Marisa: 1 to 4 is 3, 2 to 6 is 4, 3 to 8 is 5...
>
> Bárbara: Actually that is a hint of what's going on. You do plus 3, you do plus 4, plus 5, you do plus 6, you do plus 7. The number that you have to add is always one more. It's always one more. What I want to know ... That's not a rule Marisa, because if it were ... I want a rule that works in every single case and you are changing the rule. I want a rule that works in every single case that can get me from here (left column in the table shown in Fig. 2) to here (right column in the table shown in Fig. 2).

The "pattern" that Marisa described consists in adding a number to the value of the independent variable in order to get to the value for the dependent variable. For example, to get from 1 (dinner table) to 4 (people), we have to add 3; from 2 (dinner tables) to 6 (people), we have to add 4; from 3 (dinner tables) to 8 (people), we have to add 5; and so on.

Number of tables	Seating
1	+3 → 4
2	+4 → 6
3	+5 → 8
4	+6 → 10
5	+7 → 12
6	+8 → 14
50	→ 102
60	→ 122
100	→ 202
x	→ *x . 2+2*

Fig. 2. Function table presented to students in lesson 51. The first column was filled and the second was empty. Arrows have been added and the second column has been filled to illustrate Marisa's theorem-in-action.

Marisa says that every time you go down from one row to the next, you add one more than you did in the previous row to get from one column to the next. That is, the number you add to get from one column to the next increases by one each time you go down a row in the function table. At first, the research team wondered whether or not this theorem-in-action would be considered mathematically "appropriate," or correct. We wondered what made this theorem-in-action work, and whether it would work with any linear function. Our questioning was grounded in the adoption of a conceptual field framework (Vergnaud, 1982, 1988, 1994). Within this framework, an analysis and understanding of the contents of knowledge and a conceptual analysis of the domain are considered essential towards developing an understanding of children's cognitive development. At the same time, conceptual fields allow researchers and educators:

> To understand the filiations and jumps in students' acquisition of knowledge. A single concept usually develops not in isolation but in relationship with other concepts, through several kinds of problems and with the help of several wordings and symbolisms. (Vergnaud, 1988, pp. 141–142)

While we were expecting scalar and functional theorems-in-action among the children's responses, we did not want to assume that Marisa's approach was necessarily incorrect or mathematically inadequate. The Theory of Conceptual Fields allowed us to frame our analysis of Marisa's responses.

This was not the only time that Marisa used this theorem-in-action, and she was not the only one in the class to use this theorem-in-action. Marisa used this theorem-in-action during lesson 35, during the review lesson 51, and during an individual interview held in June 2004 with the first author of this paper. During lesson 35, Hannah and Aja, two other girls in Marisa's class, used this same theorem-in-action. As mentioned earlier, Nemirovsky (1996) also describes a seventh grader's approach, which is similar to Marisa's. Damian, the seventh grader, however, focuses on the following pattern in his table: the number to the right (the maximum number of people who can be seated) is always two more than the number of tables times two. Marisa, instead, focused on the following pattern: the number that you add to get from the input to the output column is always one more than it was in the previous row. Damian's "two more" is more likely to help him generate a general formula, such as $f{:}N_0 \rightarrow R$, $f(x) = 2x + 2$. In spite of being neither the only nor the first occurrence of this kind of theorem-in-action, Marisa provides us with the opportunity to examine her way of thinking in depth, as we will show below.

5. Mathematical analysis of Marisa's theorem-in-action

As outlined above, and as detailed by Vergnaud (1982, 1988, 1994), it is fundamental to analyze the relationship between knowledge produced by the learner in particular situations and contexts, and the knowledge from the perspective of the discipline we are trying to teach them about. We need to evaluate the relationship between what they *know* and what we *want* them to learn, in order to design interventions that foster their learning beyond what they can spontaneously do. Bamberger (1991) and Confrey (1991) highlight that establishing these relationships and connections also contributes to our own reconceptualization of the discipline and to a deeper and more complex understanding of the issues and concepts at stake in the conceptual fields being studied.

Analyzing the logic of children's strategies, we can gain alternative views and perspectives regarding the concepts we are attempting to teach. In other words, we can learn about theorems-in-action, strategies, and ways of conceptualizing that we have not foreseen in an *a priori* analysis, and that may not be within the repertoire of the existing research literature. Thus, by focusing on trying to understand children's strategies, we also gain feedback regarding the design of didactical situations, such as the problems presented to students and the teacher interventions to be carried out.

In terms of Marisa's theorem-in-action, as interpreted through a mathematical lens, given any linear function:

$$f(x) = mx + b$$

defined in its natural domain, we can think of the information in each one of the rows of the function table given to Marisa as shown in Fig. 3.

In order for us to examine the characteristics of the function $g(x)$, and specifically to examine if it is a linear function, we are assuming that $f(x)$ is any linear function, as in the case we are analyzing. By analyzing these characteristics, we hope to better understand why Marisa's theorem-in-action works. We want to solve the equation:

$$x + g(x) = f(x) \quad (1)$$

for $g(x)$, where

$$f(x) = mx + b \text{ with } m, b \in \Re$$

and we are looking for a function $g(x)$ that added to the identity function:

$$i(x) = x$$

yields

$$f(x).$$

Replacing

$$f(x) = mx + b$$

in Eq. (1), we obtain equation

$$x + g(x) = mx + b. \quad (2)$$

Manipulating Eq. (2) to obtain $g(x)$:

$$g(x) = mx + b - x, \qquad g(x) = (m-1)x + b.$$

In lesson 51, Marisa points out the fact that first you add 3, then 4, then 5, and then 6. You add one more each time you go down one row in the function table. In this way, she is relating the auxiliary number that she introduced in the first

	What can we add to x in order to get mx+b?	
x ⟶		mx+b

Fig. 3. One way to think of the rows in the function table Marisa worked on.

Number of Tables	*auxiliary column*	Number of People
1	+ 3	4
2	+ 4 = 3+1	6
3	+ 5 = 4+1	8
x_0	$+ c_0$	$x_0 + c_0 = f(x_0)$
$x_0 + 1 = x_1$	$+ c_0 + 1 = c_1$	$(x_0 + 1) + (c_0 + 1) = f(x_0) + 2 = f(x_1)$
$x_1 + 1 = x_2$	$+ c_1 + 1 = c_2$	$(x_1 + 1) + (c_1 + 1) = f(x_1) + 2 = f(x_2)$

Fig. 4. Mathematical structure representing the scalar features in the auxiliary column in Marisa's theorem-in-action.

row (see Fig. 4), to the auxiliary number in the row right below it. So, in Marisa's theorem-in-action, she identifies a recursive function on the intermediate, auxiliary column added by her. This is shown in the function table in Fig. 4.

Let us assume that c_0 is such that

$$x_0 + c_0 = f(x_0).$$

In other words, c_0 is the solution to Eq. (1), previously shown:

$$x + g(x) = f(x)$$

for a particular value of x, $x = x_0$. It seems that Marisa observed in the numeric sequence of the auxiliary column that to get from one row to the row below it, it is enough to add 1 more.[7] The auxiliary column can be generated by "adding one" from one row to the next to the value in the auxiliary column.

5.1. Scalar and non-scalar features of Marisa's theorem-in-action

Why is Marisa's theorem-in-action not entirely scalar? If Marisa had adopted a scalar theorem-in-action, she could have gone down the output column (see Fig. 2) adding by twos. But in her theorem-in-action, Marisa is not adding two from one row to the next in order to get the output. She did not use repeated addition by twos in order to produce the sequence of outputs in the y column of the function table. Thus, we cannot consider her theorem-in-action as purely scalar. In addition, the amount that she is adding to the input varies each time; it is not a constant amount as in the scalar approach, further justifying a characterization of her approach as non-scalar. However, if we focus on the auxiliary column highlighted in Fig. 4, we can identify one potential constant being added, although the constant is not added directly to the output column (if it were, then we would identify Marisa's theorem-in-action as scalar); instead, the constant of 1 ("*each time you go down a row in the table, you add one more*," see Fig. 4) is added to the number in the auxiliary column to get the output.

Marisa is applying some sort of scalar approach to the sequence of numbers that have to be *added* to the input in order to get the output. There are two elements that we can identify as characteristic from a scalar approach. The first is that she mainly uses addition in this theorem-in-action; she focuses on looking for the number to be *added* to the input in order to get the output. The second element characteristic of a scalar approach is that Marisa searches for a scalar pattern in the function $g(x)$.

5.2. Functional and non-functional features of Marisa's theorem-in-action

Why is Marisa's theorem-in-action not entirely functional? Her theorem-in-action does have the intention of relating input and output "directly" by seeking the function that might describe the relationship between variables. Marisa takes into account input and output, and she comes up with a way of acting on the input in order to get the output (i.e., "*each time we go down a row we have to add to the input one more than what we added to the previous input column to get the output*"). Her establishment of a relationship between both variables can be identified as an element of the functional approach.

[7] Marisa used a similar strategy when she solved a problem in the end of year assessment implemented by the EA research team. In this problem, the input column of the linear function table she was presented with skip-counted by threes until a certain point in the table. However, this was not the case for the rest of the values in the input column of the table. In the case where the skip from one row to the next was not three, Marisa gave a wrong answer based on the counting-by-three model she had generated for the intermediate column.

Marisa developed a theorem-in-action that allows her to produce outputs within a non-scalar model. Her theorem-in-action relates inputs with outputs. While we have said above that her theorem-in-action has features of a functional approach, we would not consider this as a strictly functional approach because Marisa is not using multiplication in order to get the output and because in order for her theorem-in-action to work we have to know what was added to the input in the previous row. Therefore, a "first" step is needed in order to generate the recursive sequence of numbers to add to the input to get the corresponding output. Marisa found the first number to be added by doing $f(1) - 1$; she also calculated $f(2) - 2$, $f(3) - 3$ and so on. As can be seen, this is something that can be done having some consecutive pairs $(x, f(x))$ of the function in order to infer how it behaves. In some sense, this is a disadvantage because if Marisa wanted to extend her theorem-in-action, she would need consecutive pairs of values and these are not always available.

5.3. Limitations and potential of Marisa's theorem-in-action

One central piece in the mathematical analysis of children's strategies is the assessment of both what the theorem-in-action *allows* for and what it *does not allow* for. That is, both what are the strengths of the theorem-in-action as well as what are its limitations. Identifying the limitations or weaknesses of the theorem-in-action allows us to design ways of helping children reflect on their strategies (see Martinez et al., 2005). As we just pointed out, the disadvantage or limitation of this theorem-in-action is that in order for it to work, Marisa has to know both the inputs and the outputs. By knowing both inputs and outputs, Marisa is producing a new intermediate sequence of numbers, that we called the numbers in the auxiliary column in Fig. 4. Each one of these numbers (4, 5, and 6) is added to the input to get the output. We might also hypothesize that Marisa's theorem-in-action was conceptualized and described *a posteriori*; that is, it was not a theorem-in-action that helped her solve the problem while working on it, but a theorem-in-action that helped her to describe a relationship between variables that she had already established beforehand.

Marisa's theorem-in-action did not only present shortcomings; it could also potentially promote or enable the production of a general rule regarding the relationship between variables. In our EA lessons, general rules are usually addressed in two ways: including a last row in the function table with the letter x in the column of the input (a variable amount), and asking for a general rule to be described using natural language or algebraic notation. Within these conditions, the reason for using an extra-mathematical context is to help children's conceptualization of the relationship between the two variables (e.g., number of dinner tables and maximum number of people seated at those dinner tables). The extra-mathematical context is included to facilitate children's generation of formulas or general rules for a linear function given ordered pairs of the function. The use of these extra-mathematical contexts to encourage the generation of the structure underlying the calculation for the cases of large numbers and x in the function tables has shown to be fruitful. We have found that students use the geometrical arrangement of the dinner tables, for instance, to get at the structure of the calculation for large numbers as well as for x (see Martinez et al., 2005). In Marisa's case, she is producing an auxiliary, intermediate, sequence of numbers for which there is no supporting extra-mathematical context. We hypothesize that it would be very complex for Marisa to establish the meaning of the numbers in the auxiliary column she created, within the dinner tables context. Therefore, to come up with a formula for this auxiliary sequence of numbers could be considered just as difficult as to come up with a formula for the function:

$$f(x) = 2x + 2$$

in a purely intra-mathematical context.

Before interviewing Marisa individually and learning more about her theorem-in-action, we had hypothesized another potential limitation of her approach. This potential limitation was that when Marisa approached the cases for 50, 60, and 100 dinner tables, she would not know how to proceed. During the individual interview carried out in June 2004, Marisa first explained the most common procedure used in class (to plug in the numeric values for the number of dinner tables into the formula of the function $f(x) = 2x + 2$ in order to obtain the output values). The interviewer (the first author of this paper) asked her whether she had solved the problem using this same procedure during the class. Marisa explained:

> Marisa: I thought it was . . . one to four is three . . . two to six is four . . .
>
> Mara: Can you write?

Marisa: This is three . . . This is four . . . This is five . . . This is six . . . [Referring to the values in the imaginary auxiliary column that are added to the input column to get to the output column.] This is seven . . . This is . . . eight?

Mara: Aha. Yes. Six plus eight is fourteen . . . yes . . . mmm. . .and there? . . . mmmm . . .

Marisa: It's a thousand and two [referring to the value 1000 that she has included in one of the empty rows in the table] because the numbers are not going six, seven, eight, nine, ten, eleven . . .

Mara: So, it skips now . . . but following the same rule that . . .

Marisa: Somebody came up with . . . I don't know . . .

In the individual interview we found that when Marisa encountered a gap in the sequence of numbers presented in the function table (from 6 dinner tables to 50, for instance), her hybrid theorem-in-action did not help her to find the corresponding number of people to be seated at the dinner tables. As explained before, she could have still solved this by using multiples (finding out the difference between values in the gap in the function table), but this is not the approach adopted by Marisa. Using the function table in Fig. 2, Marisa was adding one more to the numbers in the auxiliary column, from 1 through 6, adding from 3 to 8, respectively, to each of these inputs. When she got to the 50 in the input column, she stopped because she did not have the number in the auxiliary column corresponding to 49 that would have made her theorem-in-action work. At this point, her theorem-in-action stopped working because there was a gap in the function table: from 6 to 50 in the input column. We might hypothesize that by encountering these types of shortcomings and difficulties inherent to her theorem-in-action, she might modify her hybrid theorem-in-action into a functional approach, to be able to adopt a general rule for all cases, regardless of gaps in a sequence of inputs in the function table or for the case of a variable amount such as *x*.

6. Concluding remarks

Previous research has provided ample evidence for children's ability to quantify functions that deal with direct proportions (e.g., Piaget, Grize, Szeminska, & Bang, 1968/1977). Further research has also shown street sellers with relatively little schooling using scalar approaches to compute the prices of items they sell (e.g., Nunes et al., 1993; Schliemann et al., 1998). Past research has also shown that young children prefer scalar solutions, and that these approaches are an important aspect of understanding number or quantities (Kaput & West, 1994; Ricco, 1982). However, Schliemann and Carraher (e.g., 1992; Carraher & Schliemann, 2002) have also shown that scalar solutions do not allow for broader explorations of the relationships between two variables.

Thus, the prevalence of children's scalar approaches, and both the limitations and potential of these approaches, have previously been documented and highlighted. Research carried out within the scope of the EA project here reported on has also provided examples of third-grade students' emerging understanding of functional relations (e.g., Schliemann, Carraher, et al., 2001; Schliemann, Goodrow, et al., 2001; Schliemann et al., 2007). These studies have shown that young children generate such "linear function" rules from multiple instances. Schliemann, Carraher, et al. (2001) state that:

> Students may not quickly learn to identify linear functions underlying data. Despite this, and perhaps because of this, linear functions can begin to be explored as extensions to students' work with multiplication tables. Further, even though not all third grade students will initially identify and represent the functional relationships underlying data tables, they can learn significant things in the resulting discussions and slowly work functional notation into their arsenal of representational tools. (p. 7)

Thus, there is an underlying goal within this kind of research to foster functional approaches among children. From this perspective, an approach such as Marisa's, described in this paper, is of extreme interest, as it illustrates an alternative approach to the scalar and functional approaches, as Nemirovsky (1996) previously did with an older, seventh grade student. It also highlights the possible impact – on mathematical thinking – of being exposed to a functional approach.

How can we explain Marisa's theorem-in-action, given the *expected* scalar and functional theorems-in-action? Vergnaud (1982, 1988, 1994) has described the functional theorem-in-action as more complex than the scalar. In

Marisa's case, the scalar theorem-in-action has limitations for her, as it does not describe the relationship *between* variables by connecting columns in a function table. We could hypothesize in this sense that the scalar theorem-in-action provokes in Marisa some sort of disequilibrium (Piaget, 1975/1985). As Piaget stated:

> One of the sources of progress in the development of knowledge must be sought in disequilibria as such. Disequilibria alone force the subject to go beyond his current state and strike out in new directions. It is also obvious that, while disequilibria provide an essential motivational factor, they do not always lead to progress. They do so only when they give rise to developments that surpass what has previously existed and go on to specific reequilibrations. (1975/1985, p. 10)

In Marisa's case, we hypothesize that the shortcomings of the scalar approach are the motivation to develop and construct her unexpected theorem-in-action; the shortcomings play a "triggering role" in Piaget's (1975/1985) sense. Her theorem-in-action surpasses the scalar approach, and constitutes a "new composition" (Piaget, 1975/1985). New compositions or constructions,

> May result either from spontaneous initiatives of the subject, for example, inventions, or from chance encounters with objects in the environment, for example, discoveries. (Piaget, 1975/1985, p. 32)

We can therefore think of Marisa's theorem-in-action as an invention to overcome the shortcomings of the expected scalar approach and the problems encountered in the dinner tables context. We have shown what we claim to be a hybrid approach, between the scalar and the functional, in the way that linear function tables can be conceptualized. We are not interpreting Marisa's approach as necessarily transitional. We cannot argue for a scalar-hybrid-functional sequence. As described above, for instance, at some moments Marisa used her hybrid approach, and other moments she used the functional approach, depending on the specific aspects of the problem that she was trying to deal with. Although Marisa's theorem-in-action can be seen as having a limited scope of application (see also Confrey, 1991; diSessa, Hammer, Sherin, & Kolpakowski, 1991, for other descriptions of both the affordances and shortcomings of students' limited understandings and ideas) and limited power to produce results for large numbers and for x, we consider this to be one *necessary* step in Marisa's reasoning process to make sense of this type of functional relationship. We say *necessary* from Marisa's perspective, in Piaget's (1983/1987) sense, where,

> Necessity does not emanate from objective facts, which are by their nature merely real and of variable generality and therefore subject to necessary laws to a greater or lesser extent. *They only become necessary when integrated within deductive models constructed by the subject*. (p. 136. Emphasis added)

We do not claim that it is necessary for all children to construct or use this particular theorem-in-action. It is necessary in the sense that Marisa produced these relationships trying to make sense of the problem; this theorem-in-action was necessary in her way of thinking, and it became necessary to be able to go beyond a purely scalar approach, which presented Marisa with particular shortcomings.

It is our intention to look beyond the prescribed and expected ways of looking at children's thinking and approaches. We feel encouraged by this analysis of children's responses, and their grounding from a mathematical perspective, which allows us to further our understanding of children's approaches to functions, to connect these unexpected approaches to those that are expected, and to reconsider the conceptual field that they are connected to. We hope that Marisa's quasi-scalar, quasi-functional theorem-in-action, as well as others we can describe and analyze, can form part of our repertoire of learners' *expected* approaches to linear function tables. A Piaget wisely put it,

> There exists no . . . absolute end to necessity. Any necessity remains conditional and will need to be transcended. Thus, there do not exist any apodictic judgments that are intrinsically necessary. (Piaget, 1983/1987, p. 143)

Marisa's necessary theorem-in-action will be different at any given time in her development. Similarly, our repertoire of necessary descriptions of children's approaches should be continually evolving.

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