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Arithmetic and Algebra in Early Mathematics Education

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Algebra instruction has traditionally been postponed until adolescence because of historical reasons (algebra emerged relatively recently), assumptions about psychological development ("developmental constraints" and "developmental readiness"), and data documenting the difficulties that adolescents have with algebra. Here we provide evidence that young students, aged 9–10 years, can make use of algebraic ideas and representations typically absent from the early mathematics curriculum and thought to be beyond students' reach. The data come from a 30–month longitudinal classroom study of four classrooms in a public school in Massachusetts, with students between Grades 2–4. The data help clarify the conditions under which young students can integrate algebraic concepts and representations into their thinking. It is hoped that the present findings, along with those emerging from other research groups, will provide a research basis for integrating algebra into early mathematics education.

Key words: Algebra; Children's strategies; Developmental readiness; Early algebra; Functions; Mathematics K–12

INTRODUCTION

Increasing numbers of mathematics educators, policymakers, and researchers believe that algebra should become part of the elementary education curriculum. The National Council of Teachers of Mathematics [NCTM] (2000) and a special commission of the RAND Corporation (2003) have welcomed the integration of algebra into the early mathematics curricula. These endorsements do not diminish the need for research; quite the contrary, they highlight the need for a solid research base for guiding the mathematics education community along this new venture. This article will present partial findings from an investigation of eight- to ten-year-old students' algebraic reasoning during a 2 1/2 year classroom intervention study. We

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undertook this investigation in the hopes of finding evidence that young students can learn mathematical concepts and representations often thought to lie beyond their reach.

Our approach to early algebra has been guided by the views that generalizing lies at the heart of algebraic reasoning, arithmetical operations can be viewed as functions, and algebraic notation can lend support to mathematical reasoning even among young students. We focus on algebra as a *generalized arithmetic of numbers and quantities* in which the concept of *function* assumes a major role (Carraher, Schliemann, & Schwartz, in press). We view the introduction of algebra in elementary school as a move from particular numbers and measures toward relations among sets of numbers and measures, especially functional relations.

Functions have deservedly received increasing emphasis in middle and high school theorization, research, and curricula (e.g., Dubinsky & Harel, 1992; Schwartz & Yerushalmy, 1992; Yerushalmy & Schwartz, 1993). We propose that giving functions a major role in the elementary mathematics curriculum will help facilitate the integration of algebra into the existing curriculum. Key to our proposal is the notion that addition, subtraction, multiplication, and division operations can be treated from the start as functions. This is consistent with Quine's (1987) view that "a function is an operator, or operation" (p. 72).

The idea is not to simply ascribe algebraic meaning to existing early mathematics activities, that is, to regard them as already algebraic. Existing content needs to be subtly transformed in order to bring out its algebraic character. To some extent, this transformation requires algebraic symbolism. Even in early grades, *algebraic notation can play a supportive role in learning mathematics*. Symbolic notation, number lines, function tables, and graphs are powerful tools that students can use to understand and express functional relationships across a wide variety of problem contexts. In this article, we will focus on third graders' work with number lines and algebraic expressions as they solve problems in the domain of additive structures. We provide evidence that young students can make use of algebraic ideas and representations that are typically omitted from the early mathematics curriculum and thought to be beyond their reach. Because we believe that functions offer a prime opportunity for integrating algebra into existing curricular content, we also attempt to clarify what we mean by treating operators as functions.

Before we present the results of our intervention study, we will review selected mathematical and psychological ideas relevant to the suggestion that algebra has an important role in the present-day early mathematics curriculum.

EARLY ALGEBRA FROM MATHEMATICAL AND COGNITIVE PERSPECTIVES

Discussions about early algebra tend to focus on *the nature of mathematics* and *students' learning and cognitive development*. We will review background issues along these two lines.

On the Nature of Mathematics: Are Arithmetic and Algebra Distinct Domains?

The fact that algebra emerged historically after, and as a generalization of, arithmetic suggests to many people that algebra ought to follow arithmetic in the curriculum. However obvious this claim may seem, we believe there are good reasons for thinking otherwise. Assume for the moment that arithmetic and algebra are distinct topics. For example, let us assume that arithmetic deals with operations involving particular numbers, whereas algebra deals with generalized numbers, variables, and functions. Such a distinction allows for a tidy ordering of topics in the curriculum. In elementary school, teachers can focus upon number facts, computational fluency, and word problems involving particular values. Only later are letters used to stand for any number or for sets of numbers. It is not surprising that such a sharp demarcation leads to considerable tension along the frontier of arithmetic and algebra. It is precisely for this reason that many mathematics educators (e.g., Filloy & Rojano, 1989; Herscovics & Kieran, 1980; Kieran 1985; Rojano, 1996; Sutherland & Rojano, 1993) have drawn so much attention to the supposed transition between arithmetic and algebra—a transition thought to occur during a period in which arithmetic is “ending” and algebra is “beginning.” Transitional or “prealgebra” approaches attempt to ameliorate the strains imposed by a rigid separation of arithmetic and algebra. However, “bridging or transitional proposals” are predicated on an impoverished view of elementary mathematics—impoverished in their postponement of mathematical generalization until the onset of algebra instruction. Students evidence difficulties in understanding algebra in their first algebra course. But there is reason to believe that their difficulties are rooted in missed opportunities and notions originated in their early mathematics instruction that must later be “undone,” such as the view that the equals sign means “yields” (e.g., Kieran, 1981).

Consider, for example, the opportunity to introduce the concept of function in the context of addition. The expression “+3” can represent not only an operation for acting on a particular number but also a relationship among a *set* of input values and a *set* of output values. One can represent the operation of adding through standard function notation, such as $f(x) = x + 3$, or mapping notation, such as $x \rightarrow x + 3$. Adding 3 is thus tantamount to $x + 3$, a function of x. Accordingly, the objects of arithmetic can then be thought of as both particular (if $n = 5$, then $n + 3 = 5 + 3 = 8$) and general ($n + 3$ represents a mapping of $\mathbf{Z}$ onto $\mathbf{Z}$). If their general nature is highlighted, word stories need not be merely about working with particular quantities but with *sets* of possible values and hence about variation and covariation. Arithmetic comprises number facts but also the general statements of which the facts are instances.

We are suggesting that arithmetic has an inherently algebraic character in that it concerns general cases and structures that can be succinctly captured in algebraic notation. We would argue that the algebraic meaning of arithmetical operations is not optional “icing on the cake” but rather an essential ingredient. In this sense, we believe that algebraic concepts and notation need to be regarded as integral to elementary mathematics.

We are not the first to suggest that algebra be viewed as an integral part of the early mathematics curriculum. Davis (1985, 1989) argued that algebra should begin in Grade 2 or 3. Vergnaud (1988) proposed that instruction in algebra or prealgebra start at the elementary school level to better prepare students to deal with the epistemological issues involved in the transition from arithmetic to algebra; his theorizing about conceptual fields provided the rationale for a mathematics education where concepts are treated as intimately interwoven instead of separate. Schoenfeld (1995), in the final report of the Algebra Initiative Colloquium Working Groups (LaCampagne, 1995), proposes that instead of appearing in isolated courses in middle or high school, algebra should pervade the curriculum. Mason (1996) has forcefully argued for a focus on generalization at the elementary school level. Lins and Gimenez (1997) noted that current mathematics curricula from K–12 provide a limited view of arithmetic. Kaput (1998) proposed algebraic reasoning across all grades as an integrating strand across the curriculum and the key for adding coherence, depth, and power to school mathematics, eliminating the late, abrupt, isolated, and superficial high school algebra courses. Similar arguments have been developed by Booth (1988), Brown and Coles (2001), Crawford (2001), Henry (2001), and Warren (2001). In keeping with researchers' and educators' calls, the NCTM, through The Algebra Working Group (NCTM, 1997) and the NCTM Standards (2000), propose that activities that will potentially nurture children's algebraic reasoning should start in the very first years of schooling.

But do young students have the capacity for learning algebraic concepts? Let us look at what research tells us about learning and cognitive development as it relates to the learning of algebra. This brief review will set the stage for the presentation and analysis of our own data.

On the Nature of Students' Learning and Cognitive Development: Claims About Developmental Constraints

When students experience pronounced difficulties in learning algebra (see, for example, Booth, 1984; Da Rocha Falcão, 1993; Filloy & Rojano, 1989; Kieran, 1981, 1989; Kuchemann, 1981; Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Sfard & Linchevsky, 1994; Steinberg, Sleeman, & Ktorza, 1990; Vergnaud, 1985; Vergnaud, Cortes, & Favre-Artigue, 1988; and Wagner, 1981), one naturally wonders whether this is due to developmental constraints or whether the students have simply not achieved the necessary preparation. (Developmental constraints are impediments to learning that are supposedly tied to insufficiently developed mental structures, schemes, and general information-processing mechanisms. They are termed "developmental" to imply that they are intimately tied to gradually emerging structures that serve a wide variety of functions in mental life.)

Developmental constraints refer to presumed restrictions in students' current cognitive competence (i.e., "Until students have reached a certain developmental level they presumably cannot understand certain things nor will they be able to do so in the near future"), not simply their performance (i.e., "they *did not use* the property"). They are associated with the expressions "(developmental) readiness" and "appropriateness." Algebra has sometimes been thought to be "developmentally inappropriate" for young learners, lying well beyond their current capabilities.

Attributions of developmental constraints have been made by Collis (1975), Filloy and Rojano (1989), Herscovics and Linchevski (1994), Kuchemann (1981), and MacGregor (2001), among others. Filloy and Rojano (1989) proposed that arithmetical thinking evolves very slowly from concrete processes into more abstract, algebraic thinking and that there is a "cut-point separating one kind of thought from the other" (p. 19). They refer to this cut as "a break in the development of operations on the unknown" (p. 19). Along the same lines, Herscovics and Linchevski (1994) proposed the existence of a *cognitive gap between arithmetic and algebra*, characterized as "the students' inability to operate spontaneously with or on the unknown" (p. 59). Although they recognized that young children routinely solve problems containing unknowns (e.g., "5 + ? = 8"), they argued that students solve such problems without having to *represent and operate on* the unknowns; instead, they simply use counting procedures or the inverse operation to produce a result. Although some (e.g., Sfard & Linchevski, 1994) have considered use of the inverse operation as evidence of early algebraic thinking, others have considered this procedure as merely prealgebraic (e.g., Boulton-Lewis et al., 1997).

Historical Support for the Idea of Developmental Constraints

Parallels between historical developments and the learning trajectories of students have provided some support to the notion that students' difficulties with algebra reflect developmental constraints. For example, researchers have used Harper's (1987) insightful analysis of the historical evolution of algebra—through rhetorical, syncopated, and symbolic stages—to frame the evolution of student algebraic competence. Sfard (1995) and Sfard and Linchevski (1994) have found connections between historical and individual developments in mathematics in their theory of reification, which attempts to clarify the psychological processes underlying the development of mathematical understanding, including algebra. Likewise, Filloy and Rojano (1989) provided historical evidence for their idea of a "cut-point" separating arithmetic from algebra and argued that something analogous to this occurs in present-day mathematics education at the level of individual thought.

Re-examining Assumptions About Young Students' Capabilities

Faced with historical analysis and empirical evidence, it would be easy to conclude that students face a long, difficult journey to algebra. However, history can be misleading. Negative numbers were the subject of heated debate among professional mathematicians less than 2 centuries ago, yet they are standard fare in curricula designed for today's preadolescent and adolescent students. This is not to deny that negative numbers are challenging for many students. In fact, many of the obstacles that students face may indeed be similar to those faced by earlier mathematicians. However, when new mathematical and scientific knowledge have been systematized and worked into the corpus of existing knowledge, it may become surprisingly approachable.

Historical developments in mathematics are important for understanding the dilemmas and difficulties students may encounter. But deciding whether certain ideas and methods from algebra are within the grasp of young students requires empirical studies with young students who have had access to activities and challenges that involve algebraic reasoning and algebraic representation. As Booth (1988) has suggested, students' difficulties with algebra may result from the limited ways that they were taught about arithmetic and elementary mathematics.

The classroom studies by Davydov's team (see Bodanskii, 1969/1991; Davydov, 1969/1991) show that Russian children who received instruction in algebraic representation of verbal problems from Grades 1 to 4 performed better than their control peers throughout later school years and showed better results in algebraic problem solving when compared to sixth and seventh graders in traditional programs of 5 years of arithmetic followed by algebra instruction from Grade 6. Other promising results concerning work on equations come from interview studies in Brazil. Brito Lima and da Rocha Falcão (1997) found that first- to sixth-grade Brazilian children can develop written representations for algebraic problems and, with help from the interviewer, solve linear equation problems using different solution strategies. Lins Lessa (1995) found that after only one instructional session, fifth-grade students (11- to 12-year-olds) could solve verbal problems or situations presented on a balance scale that corresponded to equations, such as $x + y + 70 = 2x + y + 20$ or $2x + 2y + 50 = 4x + 2y + 10$. She also showed that the children's solutions in a posttest were based on the development of written equations and, in more than 60% of the cases, the solutions were based on the use of syntactical algebraic rules for solving equations. In our own work, we have found that even 7-year-olds can handle the basic logic underlying additive transformations in equations (Schliemann, Carraher, & Brizuela, in press; Schliemann, Carraher, Brizuela, & Jones, 1998).

Evidence that elementary school children in U.S. classrooms can reason algebraically has been building up over the years as a result of reform in mathematics education that led to the introduction of discussions on generalization of number patterns in elementary school. Carpenter and Franke (2001) and Carpenter and Levi (2000) showed that fairly young children who participated in classroom activities that explore mathematical relations can understand, for instance, that "$a + b – b = a$" for any numbers a and b. Schifter (1999) described compelling examples of implicit algebraic reasoning and generalizations by elementary school children in classrooms where reasoning about mathematical relations is the focus of instruction. Blanton and Kaput (2000) further showed third graders making robust generalizations as they discuss operations on even and odd numbers and consider them as placeholders or as variables.

Another set of studies examined young children's generalizations and their understanding of variables and functions. Davis (1971–1972) and his colleagues in the Madison Project developed a series of classroom activities that could be used to introduce, among other things, concepts and notation for variables, Cartesian coordinates, and functions in elementary and middle school. These tasks were successfully piloted in Grades 5 to 9 and, as Davis stressed, many of the activities are appropriate for children from Grade 2 onward. In a previous classroom intervention study, we have found that, given the proper challenges, third graders can engage in algebraic reasoning and work with function tables, using algebra notation to represent functional relations (Brizuela, 2004; Brizuela & Lara-Roth, 2001; Brizuela, Carraher, & Schliemann, 2000; Carraher, Brizuela, & Schliemann, 2000; Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, 2001, in press; Schliemann, Goodrow, & Lara-Roth, 2001). Evidence of algebraic reasoning has been found even among first and second graders who participated in Early Algebra activities inspired by Davydov's (1975/1991) work (Dougherty, 2003; Smith, 2000). More recently, we found (Brizuela & Schliemann, 2004) that fourth graders (9- to 10-year-olds) who participated in our Early Algebra activities can use algebraic notation to solve problems of an algebraic nature.

Why Research in Early Algebra Is Still Needed

It may seem that the major issues of Early Algebra Education were settled when Davydov's team of researchers showed success in introducing algebra to young learners (Davydov, 1969/1991). We regard their work as groundbreaking, but view it as opening rather than closing the field of early algebra. It highlighted many of the means by which algebraic concepts could be made accessible and meaningful to young children, but there is still much to do. Although the Soviet work gives a straightforward look at early algebra from the teachers' perspective, it is vitally important to understand how students make sense of the issues. What are their questions? What sorts of conflicts and multiple interpretations do they generate? How do their initially iconic drawings eventually evolve into schematic diagrams and notation? What sorts of intermediary understandings do children produce?

In addition, the Davydov team has tended to downplay the potential of arithmetic as a basis for algebraic knowledge. At times, they even argue that arithmetic be introduced into the curriculum *after* algebra. The authors do make a good case for using unmeasured quantities in order to encourage students to reflect upon quantitative relations and to make it difficult for them to bypass such reflection by resorting directly to computations. However, it is difficult to conceive of children developing strong intuitions about number lines, for example, without ever having used metrics and without having a rudimentary grasp of addition and subtraction facts.

Finally, we need to have a better understanding of how functions can be introduced in arithmetical contexts. As we noted elsewhere (Carraher, Schliemann, & Brizuela, 2000), one of the best-kept secrets of early mathematics education is that addition is a function, or at least can be viewed as a function. Of course one can go a long way without considering addition a function, and this is what the traditional curriculum does. Tables of numbers can be thought of as function tables (Schliemann, Carraher, & Brizuela, 2001). But since children can fill out tables correctly without making explicit the functional dependence of the dependent variable on the independent variable(s), they may merely be extending patterns. There needs to be an additional step of making explicit the functional dependence underlying such patterns. This demands that students make generalizations in language, algebraic notation, or other representations such as graphs and diagrams.

We tried to deal with the issues above in a 30-month longitudinal study with children between 8 and 10 years of age (middle of second to end of fourth grade). We developed and examined the results of a series of activities aimed at bringing out the algebraic character of arithmetic (see Brizuela & Schliemann, 2004; Carraher, Brizuela, & Earnest, 2001; Carraher, Schliemann, & Schwartz, in press; Schliemann & Carraher, 2002; Schliemann, Carraher, Brizuela, Earnest, et al., 2003; Schliemann, Goodrow, et al., 2001). In this article, we describe the outcomes of two of the lessons we implemented in third grade. Our aim is to exemplify how young children, as they learn addition and subtraction, can be encouraged to integrate algebraic concepts and representations into their thinking.

Students are often introduced to algebra through first-order equations of the form $ax + b = cx + d$ (or a variant such as $ax + b = d$). Unfortunately, this introduces far too many new issues at once and further encourages students to view variables as having a single value. These problems can be largely avoided by giving students the opportunity to work extensively with functions before encountering equations. In our approach, they first encounter "additive offset" functions, a subclass of linear functions of the form $x + b$. Because the constant of proportionality is 1, issues of proportional growth are temporarily suppressed in order to highlight the additive constant aspect of linear functions. This has certain distinct advantages. Children's initial intuitions about order, change, and equality first arise in additive situations. And, as we will show, as children work with the number line and with a variable number line, they can come to effectively deal with variables and functional covariation to approach problems involving additive relations. Additionally, first-order equations can be finally introduced as a special condition in which two functions have been constrained to be equal.

THE CLASSROOM STUDY

Young Children Doing "Algebrafied Arithmetic"

The present data come from a longitudinal study with 69 students, in four classrooms, as they learned about algebraic relations and notation, from Grade 2 to 4. Students were from a multiethnic community (75% Latino) in the Greater Boston area. From the beginning of their second semester in second grade to the end of their fourth grade, we implemented and analyzed weekly activities in their classrooms. Each semester, students participated in six to eight activities, each activity lasting for about 90 minutes. They worked with variables, functions, algebraic notation, function tables, graphs, and equations. The algebraic activities related to addition, subtraction, multiplication, division, fractions, ratio, proportion, and negative numbers. All the activities were videotaped. Here we provide a general overview of lessons employed during Grade 3. Our sequence of activities does not constitute a fully developed Early Algebra curriculum and is included mainly to provide a context for the subsequent analysis.

During the first term of their third grade, when the children were eight and nine years of age, we held eight 90-minute weekly meetings in each of four classes, working with additive structure problems and representations. We will first provide a broad description of how the students were introduced to number lines from Lesson 3 to Lesson 6. We will then describe in more detail the activities we developed in Lesson 7 and Lesson 8 in one of the classrooms. The instructors, David and Bárbara, are coauthors in the present article. The lessons generally did not begin with a polished mathematical representation or with a problem supporting merely one correct response. Children were instead presented with an open-ended problem involving indeterminate quantities. After holding an initial discussion about the situation, we asked students to express their ideas in writing. We then discussed their representations and introduced conventional representations; the conventions we chose to introduce had connections both to the problem we were working with and to the students' own representations.

Introducing Number Lines

By the time our students had reached Lesson 7 and 8 in the fall semester, they had already spent several hours working with number lines.

Lesson 3

Their first encounter took place in Lesson 3 when we strung twine across the room to which large, easily readable numbers were attached at regular intervals; this physical number line offered their first look at negative numbers (it ranged from –10 to +20 over approximately 10 meters). It also allowed us to carry out discussions about how changes in measured and counted quantities—age, distance, money, candies, and temperature—as well as pure numbers mapped onto number line representations. For such activities, children represented diverse values by locating themselves at various positions along the physical number line. They learned, with pleasure often bordering on glee, to interpret displacements on the number cord-line in terms of their growing older, getting warmer or colder, earning and spending money (which several referred to as "wasting money," perhaps from the ambiguous Spanish verb, *gastar*[1]). The context served to support vigorous discussions about the relationships among physical quantities and the order of numbers. Considerations about debt were crucial to clarifying what negative numbers mean and for helping the students realize, for example, the difference between having $0 and having –$2 despite the fact that one's pockets were probably empty in either case.

[1] A large percentage of the students in the classes we teach come from immigrant Spanish-speaking homes.

In one class, David raised various questions about the number line ("How many numbers are there on the line? Do the numbers only begin at –10 and proceed to +20 or are there more? How far can the number line go?"). Students at first suggested that the only numbers on the line were those for which there were printed labels (–10 through +20). David asked whether those were all the numbers that existed. A student suggested that there were "50 numbers," adding "You can only go as far as the wall." David suggested that the students ignore the wall; the important thing was to make sure that all the numbers were included. From that moment, the students began to suggest locations to which the number line would extend: across the playground, to other regions of the country, and eventually outer space itself. Each time that students mentioned a new location, David asked whether all the numbers were now accounted for. Eventually several students suggested that the number line went "to infinity," and explained that it "kept going on and on."

Lesson 4

A week later, David asked the students to explain what the number line was that they had been discussing. In one classroom, a child mentioned having to behave "like ghosts" when using the number line, because one penetrated walls to reach the desired numbers. Another child referred to the number line as "a time machine": it allowed him to proceed backward and forward in time when he treated the numbers as referring to his age. One student objected, arguing that people cannot go backward in time, to which another responded, "You can in your imagination."

In a subsequent lesson, we moved from the number line made of twine to diagrams on paper and projected onto a screen from an overhead projector. We introduced arrows linking points on the number line to represent changes in values. When several arrows were connected on the number line, students learned that they could simplify the information by shortcuts that went from the tail of the first arrow to the head of the last arrow. They also learned to express such shortcuts or simplifications through notation: for example, "+7 – 10" could be represented as "–3," since each expression had the same effect. They could show the similarities and differences between the two by making trips along the number line, in front of the class or on paper. The large-scale number line, projected or strung in front of the whole class, allowed students and teacher to discuss mathematical operations in a forum where students who were momentarily not talking could nonetheless follow the reasoning of the participants. This helped students deal with a range of issues, including the immensely important one of distinguishing between numbers as points and numbers as intervals. Operands could be treated as points or as intervals (from 0 to the endpoint), but operations such as "+6" referred to the number of unit spaces between positions and not to the number of fence posts or markers, so to speak, lying between numbers.

Lesson 6

In Lesson 6, the fourth session in which we dealt with number lines, we introduced a "variable number line" as a means of talking about operations on unknowns (see Figure 1). "*N* minus 4" could be treated as the result of displacement of four spaces leftward from *N*, regardless of what number *N* stood for. With an overhead projector, we sometimes employed two number lines: the variable number line and a standard number line with an origin at 0. By placing one line over the other (they shared the same metric), students could determine the value of *N*; it was the integer aligned with *N*. They also gradually realized they could infer the values of, say, $N + 3$ from seeing that $N + 7$ sat above 4 on the regular number line. The connections to algebraic equations should be obvious to the reader.

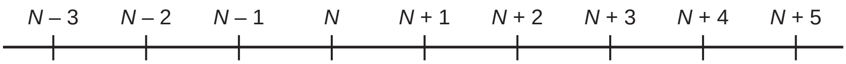

Figure 1. The *N*-number line.

Working With Unknown Quantities

Lesson 7

Figure 2 shows the problem we presented to students in Lesson 7. The problem did not specify the amounts of money that Mary and John have in their piggy banks at the beginning of the story; it merely stated that they have equal amounts. In the subsequent parts of the problem, the students learned about changes that occurred in the amounts. In the final part, the students learned how much Mary had in her piggy bank on Thursday. From this information, the students ultimately determined how much the protagonists had at the beginning and how much they had on each

Mary and John each have a piggy bank.

On *Sunday,* they both had the same amount in their piggy banks.

On *Monday*, their Grandmother comes to visit them and gives $3 to each of them.

On *Tuesday*, they go together to the bookstore. Mary spends $3 on Harry Potter's new book. John spends $5 on a 2001 calendar with dog pictures on it.

On *Wednesday*, John washes his neighbor's car and makes $4. Mary also made $4 babysitting. They run to put their money in their piggy banks.

On *Thursday,* Mary opens her piggy bank and finds that she has $9.

Figure 2. The Piggy Bank problem.

day in the story. We initially displayed the problem as a whole (except for the line on what happened on Thursday) so that the students could understand that it consisted of a number of parts. Then we covered up all days excepting Sunday.

Representing an Unknown Amount

After reading what happened each day, students worked alone or in pairs, trying to represent on paper what was described in the problem. During this time, members of the research team asked individual children to explain what they were doing and questioned them in ways that encouraged them to develop more adequate representations. In what follows we attempt to describe, on the basis of what was depicted in the videotapes and in the children's written work, the content of the classroom discussion that followed and children's insights and achievements as they attempted to represent and solve the various steps of the problem.

Sunday: After Kimberley read the Sunday part for the whole class, Bárbara, the researcher running this class, asked students if they knew how much money each of the characters in the story had. The children stated in unison, "No," and did not appear to be bothered by that. A few uttered "*N*," and Talik stated, "*N*, it's for anything." Other children shouted "any number" and "anything."

When Bárbara asked the children what they are going to show on their answer sheets for this first step in the problem, Filipe said, "You could make [*sic*] some money in them, but it has to be the same amount." Bárbara reminded him that we do not know what the amount is, and he then suggested that he could write *N to represent the unknown amount*. Jeffrey immediately said that that is what he was going to do. The children started writing in their handouts, which contained information only about Sunday and a copy of the *N*-number line. Bárbara reminded the students that they could use the *N*-number line (a number line with *N* at the origin and a metric of one unit) on their paper if they wanted. She also drew a copy of the *N*-number line on the board.

The students worked for about 3 minutes, drawing piggy banks and representing the amounts in each of them. Four children attributed specific values for Mary and John on Sunday. Five represented the amount as *N*, usually inside a drawing of a piggy bank. Two children placed a question mark inside or next to each piggy bank. And five children drew piggy banks with no indication of what each would contain.

Jennifer, one of the students who used *N* to represent the initial amount in each bank (see Figure 3), drew two piggy banks, labeling one for Mary, the other for John, and wrote next to them a large *N* after the statement "Don't know?" In a one-on-one interaction with Jennifer, David (present but not serving as instructor in the class) points to the *N* on her handout and asked:

David: So, what does it say over here?
Jennifer: *N*.
David: Why did you write that down?
Jennifer: Because you don't know. You don't know how much amount they have.
David: So, does *N* . . . What does that mean to you?

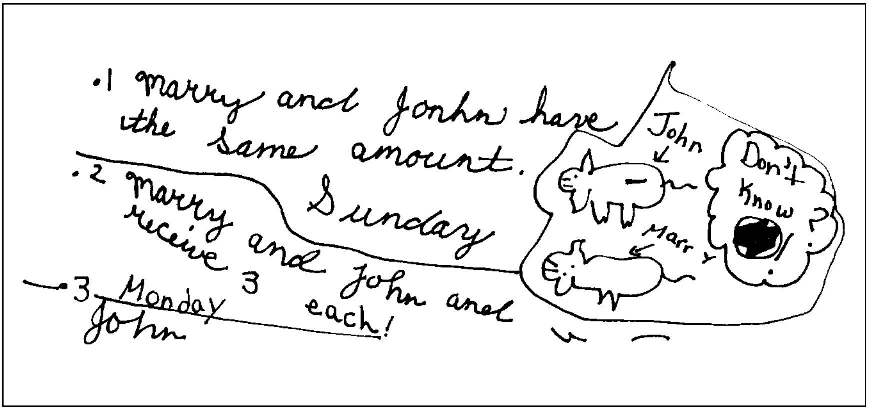

Figure 3. Jennifer's initial representation for the problem.

Jennifer:	*N* means any number.
David:	Do they each have *N*, or do they have *N* together?
Jennifer:	[Does not respond.]
David:	How much does Mary have?
Jennifer:	*N*.
David:	And how about John?
Jennifer:	*N*.
David:	Is that the same *N* or do they have different *N*s?
Jennifer:	They're the same, because it said on Sunday that they had the same amount of money.
David:	And so, if we say that John has *N*, is it that they have, like, $10 each?
Jennifer:	No.
David:	Why not?
Jennifer:	Because we don't know how much they have.

From the very beginning of this class, the children themselves proposed to use *N* to represent an unknown quantity. We had introduced the convention before in other contexts, but now it was making its way into their own repertoire of representational tools. Several children appear to be comfortable with the notation for an unknown as well as with the idea that they can work with quantities that may remain unknown.

Talking About Changes in Unknown Amounts

Monday: When the children read the statement about what happened on Monday, that is, that each child received $3 from their grandmother, they inferred that Mary and John would continue to have the same amount of money as each other, and that they both would have $3 more than the day before:

A child:	Now they have three more than the amount that they had.
Bárbara:	Do you think that John and Mary still have the same amount of money?

Children: Yeah!

Bárbara: How do you know?

Talik: Because before they had the same amount of money, plus three, they both had three more, so it's the same amount.

David: The same amount as before or the same amount as each other?

Talik: The same amount as each other. Before, it was the same amount.

David: And do they have the same amount on Monday as they had on Sunday?

Talik: No.

Another child: You don't know!

Bárbara: What is the difference between the amount they had on Sunday and the amount they had on Monday?

Children: They got three more.

Talik: Yeah. They have three more. They could have a hundred dollars; Grandma comes and gives them three more dollars, so it's a hundred and three.

Bárbara next asked the children to propose a way to show the amounts of money in the piggy banks on Monday. Nathan was the first to propose that on Monday they would each have *N* plus 3, explaining, "Because we don't know how much money they had on Sunday, and they got plus, and they got three more dollars on Monday." Talik proposed to draw a picture showing Grandma giving money to the children. Filipe represented the amount of money on Monday as "? + 3." Jeffrey said that he wrote "three more" because their Grandmother gave them three more dollars. The drawings in Figure 4 are Jeffrey's spontaneous depictions of *N* + 3. In each case, the 3 units are individually distinguished atop a quantity, *N*, of unspecified amount.

James proposed and wrote on his paper that on Sunday each would have "*N* + 2" and therefore on Monday they would have *N* + 5. It is not clear to us why he chose *N* + 2 as a starting point. Carolina wrote *N* + 3. Jennifer wrote *N* + 3 in a vertical arrangement with an explanation underneath: "3 more for each." Talik wrote *N* + 3 = *N* + 3. Carolina, Adriana, and Andy wrote *N* + 3 inside or next to each piggy

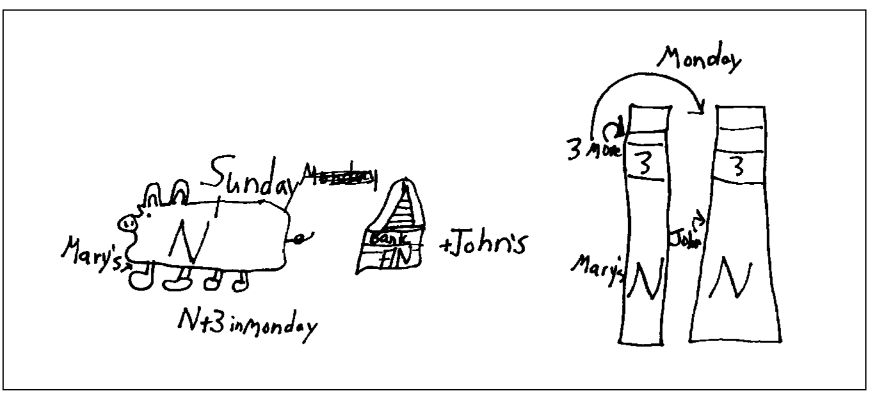

Figure 4. Jeffrey's representations for what happened on Sunday and on Monday.

bank under the heading Monday. Jeffrey wrote $N + 3$ for Monday and explained that that is because their Grandma gave them 3 dollars more. But when David asks him how much they had on Sunday, Jeffrey answered, “Zero.” Max, sitting next to him, then says, “You don’t know.” Jimmy, who first represented the amounts on Sunday as question marks, wrote $N + 3$, with connections to Mary and John’s schematic representation of piggy banks, and explained, “Because when the Grandmother came to visit them they had like, *N*. And then she gave Mary and John \$3. That’s why I say [pointing to $N + 3$ on paper] *N* plus three.”

Bárbara commented on Filipe’s use of question marks, and he and other children acknowledged that *N* is another way to show the question marks. She then told the class that some of the children proposed specific values for the amounts in the piggy banks on Sunday. Speaking against this approach, Filipe stated that “nobody knows [how much they have]” and James said that these other children “are wrong” to assign specific values. Jennifer clarified that it *could* be one of the suggested amounts.

At this point, several children seemed comfortable with the notation for an unknown and with the idea that they could work with quantities that might remain unknown. Their written work showed that, by the end of the class, 11 of the 16 children had adopted $N + 3$ for the amounts each would have on Monday. Only one of the children continued to use specific amounts in his worksheet, and four produced drawings that could not be interpreted or written work that included *N* but in incorrect ways such as $N + 3 = N$.

Operating on Unknowns With Multiple Representations

Tuesday: When they considered what happened on Tuesday, some of the children appeared puzzled and uncomfortable as they wondered whether there would be enough money in the piggy banks to allow for the purchases. A student suggested that the protagonists in the story probably had \$10. Others assumed that there must be at least \$5 in their piggy banks by the end of Monday, otherwise John could not have bought a \$5 calendar.

Bárbara recalled for the class what happened on Sunday and Monday. The children agreed that on Monday each of the children had the same amounts. She then asked these questions:

Bárbara: Adriana, what happens on Tuesday?
Adriana: On Tuesday, they had different amounts of money.
Bárbara: Why do they have different amounts of money?
Adriana: Because they spent, Mary spent \$3, and John spent \$5.
Bárbara: So, who spent more money?
Adriana: John.
Bárbara: So, on Tuesday, who has more money on Tuesday?
Adriana and other children: Mary.

Jennifer described what happened from Sunday to Tuesday and concluded that, on Tuesday, Mary ended up with the same amount of money that she had on

Sunday, "because she spends her \$3." At this point, Bárbara encouraged the children to use the N-number line to represent what has been going on from Sunday to Tuesday. Always dialoguing with the children and getting their input, she drew arrows going from N to $N + 3$ and then back to N again to show the changes in Mary's amounts. She showed the same thing with algebraic notation, narrating the changes from Sunday to Tuesday, step by step, and getting the children's input while she wrote $N + 3 – 3$. She then wrote a bracket under $+3 – 3$ and a 0 below it. She commented that $+3 – 3$ is the same as 0, and extended the notation to $N + 3 – 3 = N + 0 = N$. Jennifer then explained how the \$3 spent cancel out the \$3 given by the Grandmother: "Because you added three, right, and then she took, she spent those three and she has the number she started with."

Bárbara then led the children through John's transactions on the N-number line, drawing arrows from N to $N + 3$, then $N – 2$, for each step of her drawing. During this process she used algebraic script to register the states and transformations, and with the students' input kept track of the states and transformations, eventually writing $N + 3 – 5$ to express John's amount at the end of Tuesday. Some children suggested that this amount is equal to "N minus 2," an inference that Bárbara registered as $N + 3 – 5 = N – 2$.

Bárbara asked Jennifer to approach the number line and show the difference between John and Mary's amounts on Tuesday. Jennifer at first pointed vaguely to the interval between $N – 2$ and N. When Bárbara asked her to show exactly where the difference starts and ends, Jennifer correctly pointed to $N – 2$ and to N as the endpoints of the segment. David asked Jennifer how much one would have to give John so that he had the same amount of money as he had on Sunday. Jennifer answered that we would have to give \$2 to John and explains, showing on the number line, that if he is at $N – 2$ and we add 2, he goes back to N. Bárbara represented what Jennifer has said as $N – 2 + 2 = N$. Jennifer took the marker from Bárbara's hand, drew brackets around the expression "$–2 + 2$," and wrote 0 under it. Bárbara asked why "$–2 + 2$" equals 0 and, together with Jennifer, went through the steps corresponding to $N – 2 + 2$ on the N-number line showing how $N – 2 + 2$ ends up at N. Talik showed how this works when N has the value of 150. Bárbara used Talik's example of $N = 150$ to show how one ends up at value N on the number line.

Nathan's drawing in Figure 5 depicts Sunday (top), Monday (bottom left), and Tuesday (bottom right). For Tuesday, he first drew iconic representations of the calendar and the book next to the values \$5 and \$3, respectively, the icons and dollar values connected by an equals sign. During his discussion with an in-class interviewer, he wrote the two equations $N + 3 – 5 = N – 2$ and $N + 3 – 3 = N$, using the N-number line as support for his decisions. Later, when he learned that N was equal to 5 (after receiving the information in the problem about Thursday), he wrote 8 next to $N + 3$ on the Monday section of his worksheet.

Wednesday: Filipe read the Wednesday step of the problem. Bárbara asked whether Mary and John will end up with the same amount as on Monday. James said, "No," and Adriana then explained that Mary will have $N + 4$ and John will

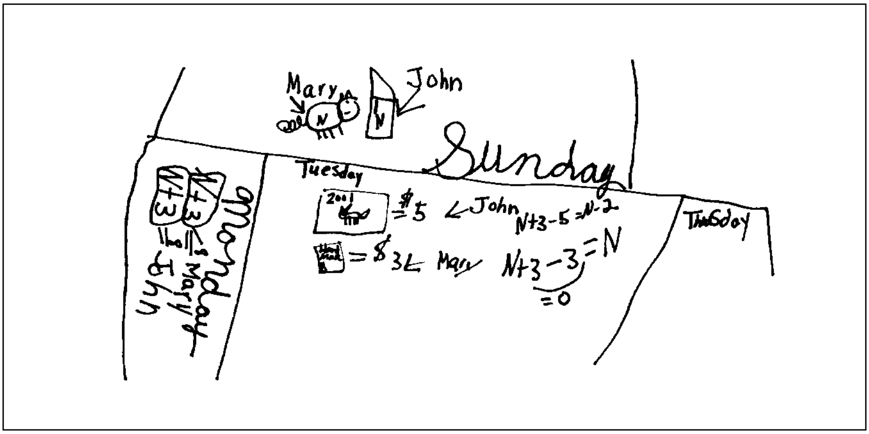

Figure 5. Nathan's representation of the problem.

have $N + 2$. Bárbara drew an N-number line and asked Adriana to tell the story using the line. Adriana represented the changes for John and for Mary on the N-number line, much as she would on a regular number line. Bárbara wrote $N + 4 = N + 4$ and then $N - 2 + 4 = N + 2$. Talik volunteered to explain this. He said that if you take 2 from the 4, it will equal 2. To clarify where the 2 comes from, Bárbara represented the following operations on a regular number line: $-2 + 4 = 2$.

Bárbara then asked if anyone could show why $N + 3 - 3 + 4$ equals $N + 4$. Talik volunteered to do so, crossed out the subexpression $+3 - 3$, and said, "We don't need that anymore." Jennifer stated that this is the same as 0. This is a significant moment, because we had not introduced the procedure of striking out the sum of a number and its additive inverse (although we *had* used brackets to simplify sums). Talik's *striking* instead of *bracketing* shows his understanding that a number and its bracketed inverse yield 0. Bárbara proposed to write out an equation conveying what happened to John's amounts throughout the week. The students helped her go through each of the steps in the story and build the equation $N + 3 - 5 + 4 = N + 2$. But they did not initially agree upon the expression for the right-hand side of the equation. Barbara helped them visualize the operations on the N-number line. She asked Jennifer to show how the equation could be simplified. While Jennifer pondered, Bárbara pointed out that this problem was harder than the former and asked her to start out with $+3 - 5$; Jennifer answered, "Minus two." Then they bracketed the second part at $-2 + 4$, and Jennifer, counting on her fingers, said it is $+2$ and wrote. Talik came up to the front and explained, "N is anything, plus 3, minus 5, is minus 2; N minus 2 plus 4, equals [while counting on his fingers] N plus 2." Talik then tried to group the numbers differently, first adding 3 and 4 and then taking away 5. Bárbara showed that when grouping $+3 + 4$, one arrives at $+7$. Next, $+7 - 5$ results in $+2$. Hence, the answer is 2, regardless of the order of the operations.

Discovering a Particular Value and Instantiating Other Values

Thursday: When Amir read the Thursday part of the problem, where it stated that Mary ended with \$9, several children suggested out loud that N has to be 5. So Bárbara asked the children, "How much does John have in his piggy bank [at the end of Thursday]?" Some students claimed that John (whose amount was represented by $N + 2$) has "two *more*," apparently meaning "two more than N." Jennifer, James, and other children said that he has 7. Some of the students apparently determined this by adding 5 + 2. Others determined it from Mary's final amount (9): Because $N + 2$ (John's amount) is 2 less than $N + 4$ (Mary's amount), John would have to have 2 less than Mary (known to have 9). Bárbara concluded the lesson by working with the students in filling out a 2×4 table displaying the amounts that Mary and John had across the 4 days. Some students suggested expressions with unknown values; others suggested using the actual values, as inferred after the information about Thursday's events had been disclosed.

A New Context: Differences Between Heights

Lesson 8

The following week, in Lesson 8 we asked the same group of students to work on the problem shown in Figure 6 (see Carraher, Schliemann, & Brizuela, 2000, in press, for a previous analysis of the same problem by another group of students). The problem states the *differences* in heights among three characters without revealing their actual heights. The heights could be thought to vary insofar as they could take on a set of possible values. Of course that was *our* view. The point of researching the issue was to see what sense the *students* made of such a problem.

After discussing each statement in the problem, the instructor encouraged the students to focus on the differences between the protagonists' heights (see Carraher, Brizuela, et al., 2001, for details on this first part of the class), and the students were

Tom is 4 inches taller than Maria.
Maria is 6 inches shorter than Leslie.
Draw Tom's height, Maria's height, and Leslie's height.
Show what the numbers 4 and 6 refer to.

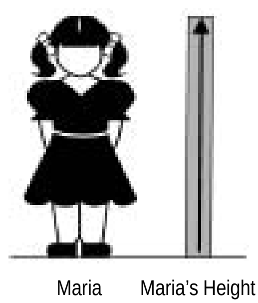

Figure 6. The Heights problem.

asked to represent the problem on individual worksheets. Most of the students used vertical lines to show the three heights (see example in Figure 7). To our surprise, one of the students (Jennifer) chose to represent the heights on a variable number line much like the one they had been working with during previous meetings (see Figure 8). Bárbara then adopted Jennifer's number line as a basis for a full-class

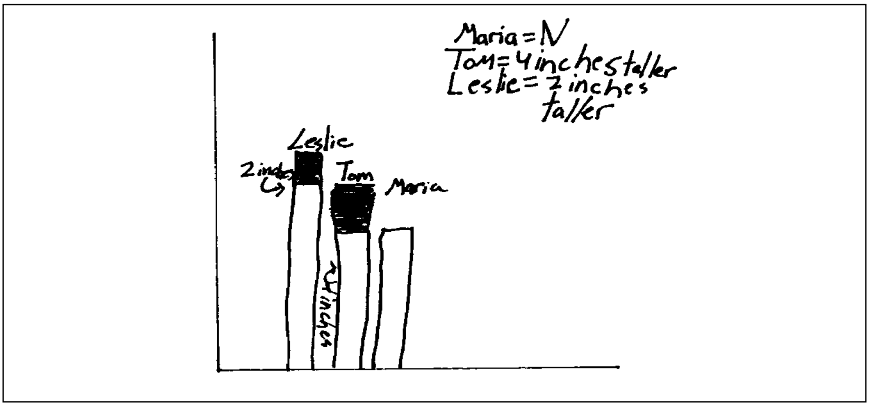

Figure 7. Jeffrey's drawing and notation for the Heights problem.

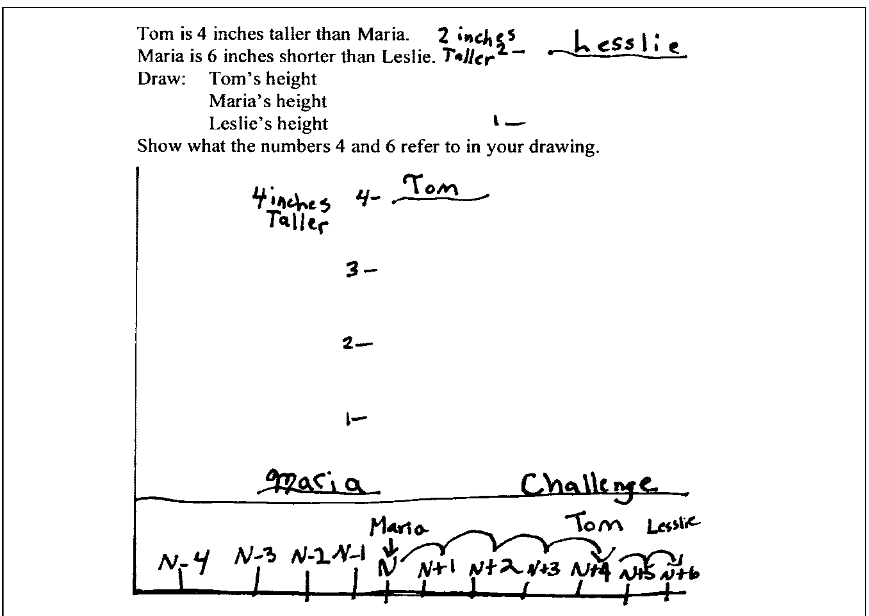

Figure 8. Jennifer's drawing (notches) showing differences but no origin. She also uses a variable number that forms the basis of subsequent discussion.

discussion of the relations among the heights. She further adopted Jennifer's assumption that Maria is located at *N* on the variable number line (see the middle number line in Figure 9).

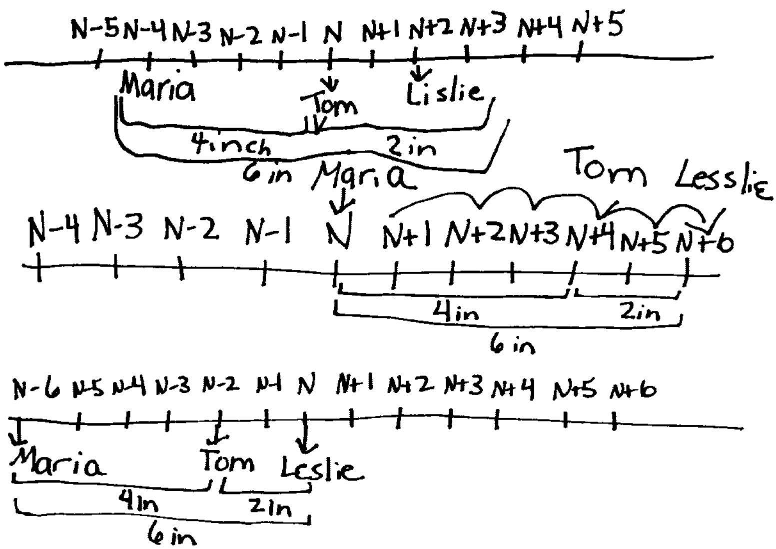

Figure 9. Three variable number line representations (on an overhead projector) used by students and teacher to discuss the cases where Maria (middle) is attributed the height of *N*; Leslie (bottom) is assigned the height of *N*; and Tom (top) is assigned a height of *N*.

Bárbara: Now if Maria's height was *N*, what would Tom's height be?
Students: *N* plus four.
Bárbara: Why?
Students: Because he would be four inches taller.
Bárbara: Mm, hmm. And what would Leslie's height be?
Nathan and students: *N* plus six.
Nathan: Because Leslie is six inches taller.

It is remarkable that Jennifer realized that a representational tool introduced in earlier classes would help to clarify the problem at hand. It is equally impressive that the remaining students appeared comfortable with this idea and easily inferred Tom and Leslie's heights (*N* + 2, *N* + 4, respectively) from Maria's (*N*). Bárbara wondered to herself whether the students realized that the decision to call Maria's height *N* was arbitrary. So she asked the students to assume instead that Leslie's height was N. The students answered that Tom's height would be $N - 2$ and Maria's would be $N - 6$ (see the bottom number line in Figure 9). They inferred this not by acting on the algebraic script but rather by making the appropriate displacements on the variable number line.

Next, Bárbara asked the students to assume that Tom's height was N. Max went to the front of the class and placed Leslie at $N + 3$ (see the top number line in Figure 9; there is an erasure under $N + 3$ where Max had first incorrectly put Leslie's name). Max realized that the difference between Tom and Leslie is 2, but nonetheless placed her 3 units to the right of Tom. (This is an example of the "fence post" issue. Students are well accustomed to the idea that a number refers to the count of elements in a set, that is, a set's cardinality. However, the issue before children often is "What should I count?" On a number line, two sorts of elements suggest themselves. One can count the number of intervals or one can count the number of "fence posts," or notches. In Max's case, he seemed to have counted the "fence posts" lying between N and $N + 3$, the delimiters.) Other students correctly stated that Leslie should be placed under $N + 2$. Finally, when Bárbara asked Amir to show where Maria's name should be located, he placed it, without hesitation, under $N - 4$.

If we focus too much on the occasional errors made by students like Max, we may fail to see the larger picture; namely, that by the end of the lesson the students are relating the given numerical differences to algebraic notation, line segment diagrams, number lines (including variable number lines), subtraction, counting, and natural language descriptions. The fluidity with which students move from one representational form to another suggests that their understanding of functions of the form $x + b$ is robust and flexible. Their willingness to use N to represent the height of any of the three characters in the story (as long as the relations among the heights of the three protagonists are kept invariant) shows an encouraging degree of robustness in their thinking.

DISCUSSION

Were These Students Doing Algebra?

Do the activities documented here qualify as algebra? Some might be tempted to argue that students had solved the Piggy Bank problem through procedures, such as "undoing," deemed merely arithmetical or prealgebraic (Boulton-Lewis et al., 1997; Filloy & Rojano, 1989). Others might note that the students did not "solve for x" in the traditional sense of applying syntactical rules to written forms, without regard to their meaning, to produce a unique value for the unknown. However, it is important to recognize what the students did achieve. They used letters to meaningfully represent variables. They used algebraic expressions such as $N + 3$ to represent functions. Furthermore, they used knowledge about the changes in quantities to formulate new algebraic expressions. They understood the relations between the daily amounts of each protagonist in the story problem; they also understood how the amounts on each day related to the starting amounts. In the discussion of the Heights problem (i.e., Lesson 8), they displayed a clear grasp of functional relations among indeterminate quantities; they were working with variables while maintaining an invariant relationship between them. And they generated appropriate expressions for the heights of the remaining protagonists regardless of which actor had been assigned the initial value *N*. As such, it appears that the students were working with functions, a fundamental object of study in algebra (Schwartz & Yerushalmy, 1994).

In several senses the lessons described above are typical of the 32 early algebra lessons we implemented during 1 semester (eight lessons implemented in each of four third-grade classrooms). At the beginning, most children relied on instantiating unknowns to particular values. But over time, in each lesson, and across the lessons, the students increasingly came to use algebraic notations and number line representations as a natural means of describing the events of problems they were presented with (see Carraher, Schliemann, & Schwartz, in press, for further analysis of this evolution in fourth grade).

Although students expressed their personal understandings in drawings and explanations, we do not suggest that their behavior was completely spontaneous. Clearly, their thinking was expressed through culturally grounded systems, including mathematical representations of various sorts. Number line representations and the use of letters to represent unknown amounts or variables are examples of cultural representations we explicitly introduced to the students. The issue is not whether they invent such representations fully on their own but rather whether they embrace them as their own—that is, whether they incorporate them into their repertoire of expressive tools.

Findings such as these have persuaded us that, given the proper experiences, children as young as eight and nine years of age can learn to comfortably use letters to represent unknown values and can operate on representations involving letters and numbers without having to instantiate them. To conclude that the sequence of operations "$N + 3 – 5 + 4$" is equal to $N + 2$, and to be able to explain, as many children were able to, that *N* plus 2 must equal 2 more than what John started out with, *whatever that value might be,* is a significant achievement—one that many people would think young children incapable of. Yet such cases were frequent and not confined to any particular kind of problem context. It would be a mistake to dismiss such advances as mere concrete solutions, unworthy of the term "algebraic." Children were able to operate on unknown values and draw inferences about these operations while fully realizing that they did not know the values of the unknowns.

We have also found further evidence that children can treat the unknowns in additive situations as having multiple possible solutions. For example, in a simple comparison problem (Carraher, Schliemann, & Schwartz, in press) wherein we described one child as having three more candies than another, our students from Grade 3 were able to propose that one child would have *N* candies and the other would have $N + 3$ candies. Furthermore, they found it perfectly reasonable to view a host of ordered pairs—(3, 6), (9, 12), (4, 7), (5, 8)—as *all* being valid solutions for the case at hand, even though they knew that in any given situation only one solution could be true. They even were able to express the pattern in a table of such pairs through statements such as "the number that comes out is always three larger than the number you start with." When children make statements of such a general nature, they are essentially talking about relations among variables and not simply unknowns restricted to single values. We have found that eight- and nine-year-old children can not only understand additive functions but also meaningfully use algebraic expressions such as "$n \rightarrow n + 3$" and "$y = x + 3$" (see Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, in press).

Cases such as those above may seem strange to people accustomed to thinking of variation in terms of changes in the values of a single entity over time. Variation is actually a broader concept than change. Sometimes, variability occurs across a set of unrelated cases, as in the variation of heights in a population or as in the covariation of heights and weights. The example of one child having three more candies than another can be understood as variation within and, even more important, invariance across a set of *possibilities*. The invariance across cases can form the basis of discussions with students about function tables containing many "solutions," one for each row. In this case, column one would correspond to the amount of candies the first child has; column two would correspond to the amount of candies the second child has. Each row contains a valid solution, insofar as it is consistent with the information given. Once students have filled out and explored such a function table, the issue is to explain it. What properties stay the same regardless of the row? This is no trivial matter, and it gives rise to general statements about number patterns that are the essence of algebra.

Such forms of variation are important because they allow one to reason both about particular values and sets of possible values. They allow one to consider unknowns as variables. This is precisely the spirit with which many students viewed the *Piggy Bank problem* before information was discovered, regarding Thursday, that finally allowed them to disregard the multiple possibilities and focus on the values to which the problem was now constrained. Students who feel the need to instantiate variables from the beginning can do so and participate in the classes from their own perspective, restricting their attention to one possible scenario from the start. This should not be a reason for concern, for we have found that such students learn from others and from class discussions, and within a few weeks they comfortably welcome algebraic representations into their own personal repertoire of expressive tools. Were their initial reliance upon instantiation due to developmental constraints, the relatively quick learning we witnessed from the piggy bank lesson to the lesson on heights could not have taken place.

The students demonstrated that they had begun to handle fundamental algebraic concepts. Still, there is much for them to learn. Algebra is a vast domain of mathematics, and the progress shown by students in the present study is the beginning of a long trajectory. In this initial stage, students benefit immensely from working in rich problem contexts that they use to help structure and check their solutions. As they become increasingly fluent in algebra, they will be able to rely relatively less on the semantics of the problem situation to solve problems. Algebraic expressions will not only capture but, more and more, will help guide their thinking and problem solving. With time, they will hopefully be able to derive valid inferences by acting on the written and graphical forms themselves, without having to reflect back on the rich problem contexts in order to successfully proceed; that is, their semantically driven problem solving will become increasingly driven by the syntax of the written expressions.

Concluding Remarks

Our work has been guided by the ideas that (1) children's understanding of additive structures provides a fruitful point of departure for an "algebrafied arithmetic"; (2) additive structures require that children develop an early awareness of negative numbers and quantities and their representation in number lines; (3) multiple problems and representations for handling unknowns and variables, including algebraic notation itself, can and should become part of children's repertoires as early as possible; and (4) meaning and children's spontaneous notations should provide a footing for syntactical structures during initial learning, even though syntactical reasoning should become relatively autonomous over time.

There may be many reasons for viewing algebra as more advanced than arithmetic and therefore placing it after arithmetic in the mathematics curriculum. But there are more compelling reasons for introducing algebra as an integral part of early mathematics. There are good reasons for considering the abstract and concrete as interwoven rather than fully distinct (Carraher & Schliemann, 2002). Addition and subtraction, multiplication and division are operations, but they are also functions and so are amenable to description through algebraic notation. If we dwell too much on the concrete nature of arithmetic, we run the risk of offering students a superficial view of mathematics and of discouraging their attempts to generalize. Although computational fluency is important (even crucial) for allowing students to reason algebraically, it does not assure that students will be attentive to the patterns underlying arithmetic and arithmetical relations. Algebraic notation (as well as tables, number lines, and graphs) offers a means for expressing such patterns clearly and succinctly. If introduced in meaningful ways, it offers the virtue of bringing together ideas that otherwise might remain fragmented and isolated.

Many have argued that young children are incapable of learning algebra because they do not have the cognitive wherewithal to handle concepts such as variables and functions (Collis, 1975; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Kuchemann, 1981; MacGregor, 2001). Our classroom studies suggest that children can handle algebraic concepts and use algebraic notation somewhat earlier than commonly supposed. There may be no need for algebra education to wait a supposed "transition period" after arithmetic. As others have shown (Carpenter & Franke, 2001; Carpenter & Levi, 2000; Schifter, 1999), there are many opportunities for introducing algebraic concepts into the curriculum during the early years of mathematics education. Moreover, we also show, as Bodanskii (1991) and Davis (1971–1972) did before us, that it is possible to introduce algebraic notation in the early grades. Our data further expand our understanding of how young children come to appropriate algebra notation as they represent open-ended problems, and we provide examples of how the teaching and learning of arithmetic operations can be related to functions.

Lest we give the mistaken impression that any mathematical concept can be learned at any time, let us set the record straight. By arguing that the algebraic character of arithmetic deserves a place in early mathematics education, we are not denying the developmental nature of mathematical skills. Number concepts, the ability to use algebraic notation, to interpret graphs, model situations, and so forth, develop over the course of many years. Even in so "simple" an area as additive structures, children need to be able to reify differences so that they can be treated as bona fide quantities with their own properties and subject to arithmetical operations. Children who are just beginning to work with addition and subtraction may interpret a statement such as "Tom is 4 inches taller than Maria, and Maria is 6 inches shorter than Leslie" as meaning that one of the children is 4 inches tall while another is 6 inches tall (Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, in press). They may confuse a height with a difference between two heights. When children get beyond this issue, this does not signify that they will no longer have troubles with additive differences. When one changes the context to one about money or introduces a number line, new problems arise (for example, "How does a difference in heights manifest itself when two line segments are used to represent people's heights?"). Thompson (1993) found that fifth-grade students may confuse second-order additive differences ("the differences between the heights of two brother-sister pairs") with first-order differences ("the differences between a brother and sister"). Issues involving concepts as rich as additive differences, ratio and proportion, division, and so on, crop up again and again in the course of one's life, and it would be naïve to assume that the challenges are conquered, once and for all, at a particular moment in time, least of all, when one learns how to perform calculations with addition and subtraction in early schooling. One could take a pessimistic view of such a conclusion: We will never cease to stumble when confronted with variations of mathematical problems that we have encountered before. But this same situation provides reason for hope, for it signifies that the schemes that have begun their evolution very early in life, perhaps as early as when a baby begins to play with nesting cups, will later prove useful to tasks that they were never designed to handle but which nonetheless succumb to metaphorical opportunism.

Early Algebra Education is by no means a well-understood field. Surprisingly little is known about children's ability to make mathematical generalizations and to use algebraic notation. As far as we can tell, at the present moment, not a single major textbook in the English language offers a coherent algebrafied vision of early mathematics. We view algebrafied arithmetic as an exciting proposition, but one for which the ramifications can only be known if a significant number of people undertake systematic teaching experiments and research. It may take a long time for teacher education programs to adjust to the fact that the times have changed.

We hope that the mathematics education community and its sources of funding recognize the importance of this venture.

REFERENCES

Blanton, M., & Kaput, J. (2000). Generalizing and progressively formalizing in a third grade mathematics classroom: Conversations about even and odd numbers. In M. Fernández (Ed.), *Proceedings of the Twenty-second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 115–119). Columbus, OH: ERIC Clearinghouse.

Bodanskii, F. (1969/1991). The formation of an algebraic method of problem solving in primary school children. In V. Davydov (Ed.), *Soviet studies in mathematics education: Vol. 6. Psychological abilities of primary school children in learning mathematics* (pp. 275–338). Reston, VA: National Council of Teachers of Mathematics.

Booth, L. R. (1984). *Algebra: Children's strategies and errors*. Windsor, UK: NFER-Nelson.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K–12* (1988 yearbook) (pp. 20–32). Reston, VA: National Council of Teachers of Mathematics.

Boulton-Lewis, G., Cooper, T., Atweh, B., Pillay, H., Wills, L., & Mutch, S. (1997). The transition from arithmetic to algebra: A cognitive perspective. In E. Pehkonen (Ed.), *Proceedings of the Twenty-first International Conference for the Psychology of Mathematics Education*: *Vol. 2* (pp. 185–192). Lahti, Finland.

Brito Lima, A. P., & da Rocha Falcão, J. T. (1997). Early development of algebraic representation among 6–13-year-old children: The importance of didactic contract. In E. Pehkonen (Ed.), *Proceedings of the Twenty-first International Conference for the Psychology of Mathematics Education, Vol. 2* (pp. 201–208). Lahti, Finland.

Brizuela, B. M. (2004). Mathematical development in young children: Exploring notations. New York: Teachers College Press.

Brizuela, B. M., & Lara-Roth, S. (2001). Additive relations and function tables. *Journal of Mathematical Behavior, 20,* 309–319.

Brizuela, B., Carraher, D. W., & Schliemann, A. D. (2000, April). *Mathematical notation to support and further reasoning ("to help me think of something")*. Paper presented at the Annual Research Presession of the National Council of Teachers of Mathematics, Chicago.

Brizuela, B. M., & Schliemann, A. D. (2004). Ten-year-old students solving linear equations. *For the Learning of Mathematics, 24*(2), 33–40.

Brown, L., & Coles, A. (2001). Natural algebraic activity. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 120–127). Melbourne, Australia: University of Melbourne Press.

Carpenter, T., & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 155–162). Melbourne, Australia: University of Melbourne Press.

Carpenter, T. P., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades.* (Res. Rep. 00–2). Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Carraher, D. W., Brizuela, B. M., & Earnest, D. (2001). The reification of additive differences in early algebra. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 163–170). Melbourne, Australia: University of Melbourne Press.

Carraher, D. W., Schliemann, A. D., & Brizuela, B. (2000). Early algebra, early arithmetic: Treating operations as functions. Plenary address presented at the *Twenty-second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Tucson, Arizona.

Carraher, D. W., Schliemann, A. D., & Brizuela, B. M. (2001). Can students operate on unknowns? In M. v. d. Heuvel-Panhuizen (Ed.), *Proceedings of the Twenty-fifth Conference of the International*

Group for the Psychology of Mathematics Education: *Vol. 1* (pp. 130–140). Utrecht, The Netherlands: Freudenthal Institute.

Carraher, D. W., Schliemann, A. D., & Brizuela, B. (in press). Treating operations of arithmetic as functions. In D. W. Carraher & R. Nemirovsky (Eds.), *Videopapers in mathematics education research*. CD-ROM issue of *monograph of the* Journal for Research in Mathematics Education. Reston, VA: National Council of Teachers of Mathematics.

Carraher, D. W., & Schliemann, A. D. (2002). Is everyday mathematics truly relevant to mathematics education? In M. E. Brenner & J. N. Moschkovich (Eds.), *Everyday and academic mathematics in the classroom, monograph of the* Journal for Research in Mathematics Education: *Vol. 11* (pp. 131–153). Reston, VA: National Council of Teachers of Mathematics.

Carraher, D. W., Schliemann, A. D., & Schwartz, J. (in press). Early algebra is not the same as algebra early. In J. Kaput, D. W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades*. Mahwah, NJ: Erlbaum.

Collis, K. (1975). *The development of formal reasoning.* Newcastle, Australia: University of Newcastle Press.

Crawford, A. R. (2001). Developing algebraic thinking: Past, present, and future. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 192–198). Melbourne, Australia: University of Melbourne Press.

Da Rocha Falcão, J. (1993). A álgebra como ferramenta de representação e resolução de problemas. In A. Schliemann, D. W. Carraher, A. Spinillo, L. Meira, & J. Da Rocha Falcão (Eds.), *Estudos em psicologia da educação matemática* (pp. 85–107). Recife, Brazil: Editora Universitária UFPE.

Davis, R. (1971–72). Observing children's mathematical behavior as a foundation for curriculum planning. *Journal of Children's Mathematical Behavior*, *1*(1), 7–59.

Davis, R. (1985). ICME-5 Report: Algebraic thinking in the early grades. *Journal of Mathematical Behavior, 4*, 195–208.

Davis, R. (1989). Theoretical considerations: Research studies in how humans think about algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra: Vol. 4* (pp. 266–274). Reston, VA: National Council of Teachers of Mathematics/Erlbaum.

Davydov, V. (Ed.). (1969/1991). *Soviet studies in mathematics education, Vol. 6. Psychological abilities of primary school children in learning mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Davydov, V. (1975/1991). Logical and psychological problems of elementary mathematics as an academic subject. *Soviet studies in mathematics education, Vol. 7. Children's capacity for learning mathematics* (pp. 55–107). Reston, VA: National Council of Teachers of Mathematics.

Dougherty, B. J. (2003). Voyaging from theory to practice in learning: Measure-up. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA, Vol. 1* (pp. 17–30). Honolulu, HI: CRDG, College of Education, University of Hawai'i.

Dubinsky, E., & Harel, G. (1992). *The concept of function: Aspects of epistemology and pedagogy*. Washington, DC: Mathematical Association of America.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 9*(2), 19–25.

Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics, 18,* 75–90.

Henry, V. J. (2001). An examination of educational practices and assumptions regarding algebra instruction in the United States. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 296–304). Melbourne, Australia: University of Melbourne Press.

Herscovics, N., & Kieran, C. (1980). Constructing meaning for the concept of equation. *Mathematics Teacher, 73*, 572–580.

Herscovics, N., & Linchevski, L. (1994) A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27,* 59–78.

Kaput, J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum. In National Council of Teachers of Mathematics (Eds.), *The nature and role of algebra in the K–14 curriculum*. Washington, DC: National Academy Press.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra, Vol. 4* (pp. 33–56). Reston, VA: National Council of Teachers of Mathematics/Erlbaum.

Kuchemann, D. E. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics* (pp. 102–119). London: Murray.

LaCampagne, C. B. (1995). *The Algebra Initiative Colloquium, Vol. 2. Working group papers*. Washington, DC: U.S. Department of Education, OERI.

Lins, R. C., & Gimenez, J. (1997) *Perspectivas em arithmética e álgebra para o século XXI.* Campinas, Brazil: Papirus.

Lins Lessa, M. M. (1995). *A balança de dois pratos versus problemas verbais na iniciação à algebra.* Unpublished master's thesis, Mestrado em Psicologia, Universidade Federal de Pernambuco, Recife, Brazil.

MacGregor, M. (2001). Does learning algebra benefit most people? In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference: The future of the teaching and learning of algebra: Vol. 1* (pp. 296–304). Melbourne, Australia: University of Melbourne Press.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 65–86). Dordrecht, The Netherlands: Kluwer Academic Publishers.

National Council of Teachers of Mathematics. (1997). *A framework for constructing a vision of algebra. Algebra working group document.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Quine, W. V. (1987). *Quiddities: An intermittently philosophical dictionary*. Cambridge, MA: Belknap Press of Harvard University Press.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: Author.

Resnick, L., Cauzinille-Marmeche, E., & Mathieu, J. (1987). Understanding algebra. In J. Sloboda & D. Rogers (Eds.), *Cognitive processes in mathematics* (pp. 169–203). Oxford: Clarendon Press.

Rojano, T. (1996). Developing algebraic aspects of problem solving within a spresadsheet environment. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra* (pp. 137–145). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking, grades K through 6. In L. Stiff & F. Curio, (Eds.), *Mathematical reasoning, K–12* (1999 yearbook) (pp. 62–81). Reston, VA: National Council of Teachers of Mathematics.

Schliemann, A. D., & Carraher, D. W. (2002). The evolution of mathematical understanding: Everyday versus idealized reasoning. *Developmental Review, 22*, 242–266.

Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (2001). When tables become function tables. In M. v. d. Heuvel-Panhuizen (Ed.), *Proceedings of the Twenty-fifth Conference of the International Group for the Psychology of Mathematics Education*: *Vol. 4* (pp. 145–152). Utrecht, The Netherlands: Freudenthal Institute.

Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (in press). *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice*. Mahwah, NJ: Erlbaum.

Schliemann, A. D., Carraher, D. W., Brizuela, B., & Jones, W. (1998, May). Solving algebra problems before algebra instruction. Paper presented at the Second Meeting of the Early Algebra Research Group, Medford, MA.

Schliemann, A. D., Carraher, D. W., Brizuela, B. M., Earnest, D., Goodrow, A., Lara-Roth, S., et al. (2003). Algebra in elementary school. In N. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA: Vol. 4* (pp. 127–134). Honolulu, HI: CRDG, College of Education, University of Hawai'i.

Schliemann, A. D., Goodrow, A., & Lara-Roth, S. (2001). Tables as multiplicative function tables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study*

Conference: The future of the teaching and learning of algebra: Vol. 2 (pp. 531–540). Melbourne, Australia: University of Melbourne Press.

Schoenfeld, A. (1995). Report of Working Group 1. In LaCampagne, C. B. (Ed.), *The Algebra Initiative Colloquium, Vol. 2: Working group papers* (pp. 11–18). Washington, DC: U.S. Department of Education, OERI.

Schwartz, J., & Yerushalmy, M. (1994). *On the need for a bridging language for mathematical modeling*. Cambridge, MA: Harvard Graduate School of Education.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14,* 15–39.

Sfard, A., & Linchevsky, L. (1994). The gains and the pitfalls of reification—The case of algebra. *Educational Studies in Mathematics, 26*, 191–228.

Smith, S. (2000). Second graders discoveries of algebraic generalizations. In M. Fernández (Ed.) *Proceedings of the Twenty-second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (pp. 133–139). Columbus, OH: ERIC Clearinghouse.

Steinberg, R., Sleeman, D., & Ktorza, D. (1990). Algebra students knowledge of equivalence of equations. *Journal for Research in Mathematics Education, 22,* 112–121.

Sutherland, R., & Rojano, T. (1993). A spreadsheet approach to solving algebra problems. *Journal of Mathematical Behavior*, *12*, 353–383.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25,* 165–208.

Vergnaud, G. (1985). Understanding mathematics at the secondary-school level. In A. Bell, B. Low, & J. Kilpatrick (Eds.), *Theory, Research & Practice in Mathematical Education* (pp. 27–45). Nottingham, UK: Shell Center for Mathematical Education.

Vergnaud, G. (1988). Long terme et court terme dans l'apprentissage de l'algebre. In C. Laborde (Ed.), *Actes du premier colloque franco-allemand de didactique des mathematiques et de l'informatique* (pp. 189–199). Paris: La Pensée Sauvage.

Vergnaud, G., Cortes, A., & Favre-Artigue, P. (1988). Introduction de l'algèbre auprès des débutants faibles: problèmes épistemologiques et didactiques. In G. Vergnaud, G. Brousseau, & M. Hulin (Eds.), *Didactique et acquisitions des connaissances scientifiques: Actes du Colloque de Sèvres* (pp. 259–280). Sèvres, France: La Pensée Sauvage.

Wagner, S. (1981). Conservation of equation and function under transformations of variable. *Journal for Research in Mathematics Education, 12,* 107–118.

Warren, E. (2001). Algebraic understanding: The importance of learning in the early years. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the Twelfth ICMI Study Conference, Vol. 2. The future of the teaching and learning of algebra* (pp. 633–640). Melbourne, Australia: University of Melbourne Press.

Yerushalmy, M., & Schwartz, J. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 41–68). Hillsdale, NJ: Erlbaum.

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