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EQUATIONS IN ELEMENTARY SCHOOL

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Although it is generally recognized that algebra has a role to play in the early mathematics curriculum, the issue of whether elementary school children are developmentally ready to use algebraic notation and to understand the syntactical rules of using algebra for solving equations is still a matter of debate. We examine, at the end of a third- to fifth-grade (8 to 11 years) intervention using a functional approach to algebra, how students try to solve word problems they represent as equations containing the same single variable on each side of the equals sign.

INTRODUCTION

We describe how elementary school students solve a word problem through an equation with variables on both sides of the equals sign, at the end of a three-year early algebra classroom intervention program.

The difficulties middle and high school students have with algebra often arise from their interpretation of the equals sign, the meaning of literals and algebraic expressions such as "3*a* +7", and the presence of variables on each side of the equals sign (see review by Kieran, 1985). Some have attributed their difficulties, which frequently mirror stumbling blocks in history of mathematics (e.g., Collis, 1975; Filloy, Puig, & Rojano, 2008; Sfard, 1995), to constraints of cognitive development. Others have suggested that the difficulties stem from the computational focus of elementary mathematics as presently conceived (Davydov, 1991; Booth, 1988; Carraher, Schliemann, & Brizuela, 2000; Kaput, 1998; Schliemann, Carraher, & Brizuela, 2007; Schoenfeld, 1995).

There is considerable evidence (see reviews by Carraher & Schliemann, 2007 and by Rivera, 2006) that elementary school children can understand the logic underlying the symbolic manipulation of equations, even before they are familiar with algebraic notation. For example, they may understand that the same transformations applied to equal quantities result in still equal quantities (see interview studies in Schliemann, Carraher, & Brizuela, 2007). However it remains unclear, with few exceptions (e.g. Bodanskii's, 1991; Brizuela & Schliemann, 2004), whether young children can learn to solve word problems by means of written equations with a single variable on each side of the equals sign. We suspect that students can learn to understand the syntax for solving equations if they are given the opportunity to examine and compare quantities at various moments during the equation solving process.

We report on partial results of a third- to fifth-grade intervention with 22 students in two classrooms, where we adopted a functional approach to algebra, based on

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understanding relations between quantities and numbers. (The study was supported by grant #0310171 from the National Science Foundation.) These results complement our previous analysis of their work with variables and functions (Schliemann, Carraher, & Brizuela, 2012). Here we describe how, at the end of fifth grade, students produced, solved, and interpreted an equation with a variable on each side of the equals sign.

Our approach

Equations may not constitute the ideal entryway into algebra insofar as a variable can easily be misunderstood as standing for a single missing value, rather than a huge, possibly infinitely large, set of values (the domain). For this and other reasons, the concept of function may provide a more suitable pathway to algebra (e.g., Schwartz & Yerushalmy, 1992). Adopting a functional approach, we view algebra in elementary and middle school as a generalized arithmetic of numbers and quantities and the introduction of algebraic activities as a move from computations on particular numbers and measures towards thinking about relations among sets of numbers and variables (Carraher, Schliemann, & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007).

Traditionally, instruction on equations aims at "solving for x," where x stands for an unknown number. But when equations are introduced in the wake of functions, the expressions on each side of the equals sign can be viewed as functions that vary over all values of x from the domain (even though many of them produce untrue statements). For example, $5 + x = 8$ sets equal the functions $f(x) = 5 + x$ and $g(x) = 8$. There is only one solution to the equation; all other values of x lead to syntactically correct but false statements.

A functional approach to equations promotes the shift of students' attention from particular (e.g. the view that x has a determined value) to the much larger and less tangible set of possible cases. At first, natural language, graphs of events, and some combination of language and tables serve as the media for representing variables and expressing generalizations. Even though these means are typically less succinct than algebraic notation, because they are relatively more accessible to young learners, algebraic notation can initially piggyback on them and on the meaning of the situations being modelled.

Our pedagogical approach, rooted in constructivist theories of learning and cognitive development, also attributes a central role to instruction and access to new representational tools and procedures (Carraher & Schliemann, 2002). Because young students typically do not draw mathematical inferences directly from algebraic expressions, elementary school instruction needs to be grounded in familiar contexts and extra-mathematical situations. There is reason to believe that, within a functional approach to early algebra, students may be able to make headway if the symbols are related to the quantities and actions regarding the situation at hand.

To evaluate the impact of a functional approach to early algebra on students' work with equations, we will briefly describe the main activities implemented in an early algebra longitudinal intervention. This will be followed by the analysis of students' ways of solving a verbal problem by representing the problem as an equation.

The intervention

The three-year intervention was carried out by researchers at an inclusion school, that is, a school where children with disabilities are placed in regular classrooms, in Boston, MA. The lessons highlighted the tension between algebra as a means for expressing properties and relations about worldly situations, on the one hand, and as a closed symbol system used to make statements about idealized mathematical objects, on the other. The lessons in third and fourth grades focused on functions and their representation through natural language, function tables, graphs, and algebraic notation. Students gradually learned to use letters to stand for unknown amounts and to establish correspondences across various representations of a given problem (Brizuela & Earnest, 2007; Carraher, Schliemann, & Schwartz, 2007; Schliemann et al., 2003). Algebraic notation was introduced first to help them express what they already knew and later to allow them to derive new insights (Brizuela & Schliemann, 2004; Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, 2007; Kaput, Blanton, and Moreno, 2007). Later, we guided students to derive conclusions directly from mathematical representations such as graphs or equations and, ultimately, to solve equations using the manipulation rules of algebra.

In fifth grade, the researchers implemented ten 90-minute lessons, including discussion and embedded assessments, on the representation of verbal problems as equations. Each lesson was followed by a 45-minute homework review session by the students' regular teacher. The following is a brief summary of the lessons:

Lesson 1 covered the representation of statements in a problem as two functions: "Anna went to the arcade with some money. She first spent five dollars playing video games. Then she won a prize where they doubled her money. Bobby went to the arcade with ten dollars. Then his mother gave him thirty more dollars. Afterwards, he spent half of all of his money playing video games. Represent Anna's and Bobby's money at the arcade at the end of the day."

Lesson 2 entailed representing and solving a problem involving a similar situation. The students used the structure shown in Figure 1, which called for operations to be noted in the oval shaped areas and results in the rectangles. This template was adapted for different equations produced by the students throughout the lessons.

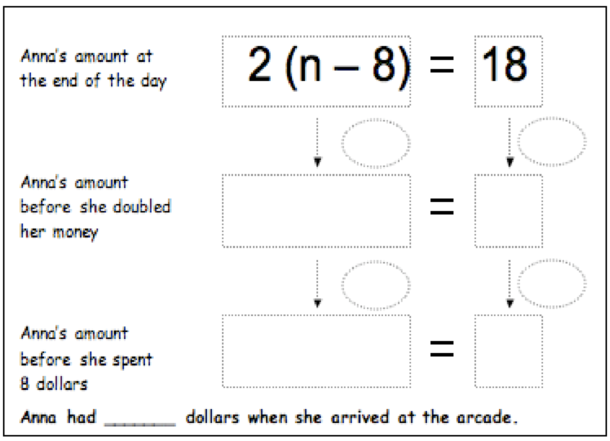

Figure 1: The template for solving equations

In lesson 3, students were given loose candies and unknown amounts of candies in closed containers. They were asked to determine how many candies were in each tube after being told that the number of candies in 3 boxes, 1 tube, and 6 candies was equal to the number in 1 box, 1 tube, and 20 candies [$3x + y + 6 = x + y + 20$].

During *lessons 4 and 5* students answered a written assessment and discussed their answers to the assessment.

In *lessons 6 and 7*, starting with a solution [e.g., $x = 5$], students produced new equations by proposing equal changes to x and to 5, checking at each step if replacing x with 5 led to same results on both sides.

Lesson 8 aimed at proficiency in solving equations with students solving a series of equations with variables on both sides of the equals sign.

During *lesson 9* the students represented and solved a verbal problem leading to an equation with a variable on both sides of the equal sign [$x+40 = 5x$] and related the solution to the story context.

In *Lesson 10* the students were given a second assessment.

RESULTS

We analyse students' attempts to represent and solve the following problem at the end of the school year, more than eight weeks after the 10 lessons on equations we described, using a template similar to the one in Figure 1:

> Claudia and Adam have been playing with numbers. They each created a rule for changing any positive number you give them. Claudia's rule: I triple the number and then add 5. Adam's rule: I double the number and then add 12 to it. Write their rules with algebra. Write an equation [in the template] that shows that Claudia's rule would give the same number as Adam's rule and solve the equation. Explain what the solution means.

Written Assessment Results

Fifteen students (68.2% of the 22 students) correctly represented Claudia's rule ($3 \times n + 5$), Adam's rule ($2 \times n + 12$), and the equation $3 \times n + 5 = 2 \times n + 12$. Three students produced operational rules like $\times\ 3 + 5$, with no explicit placeholders for variables. One student wrote $3 + 5$ and $2 + 12$, and three students produced unclassifiable expressions.

Ten students (45.5%) correctly solved the equation using the rules of algebra. Only one student attempted to apply different operations to the two sides of the equation. Of the five students who correctly generated the equation but did not solve it, four proposed equal operations for each side of the equation, but failed to correctly implement them.

Interview Results

A week after the assessment, in the individual interview, students were asked to discuss their answers to the problem and, if they had failed to do so before, to solve the equation and to explain what the solution $n = 7$ meant.

Solving the equation: The ten students who had solved the problem in the written assessment were again successful in the interview setting; eight additional students now achieved the correct solution, increasing the number of correct answers to 18 (82%). Because we viewed the interviews as further opportunities for participants in the study to learn, the initial answer provided by each student to any of the questions was registered but the interviewer asked further questions which could help students produce more advanced answers. In the interview, one of the eight students solved the equation with no help from the interviewer, four needed a small amount of support, and three needed substantial support. The categorization in terms of a small or substantial amount of support was determined by two researchers who separately examined the videotaped interviews, with discrepancies between the two evaluations resolved through discussion with the research group.

The meaning of the solution: Out of the 18 students who solved the equation in the interview, nine (50%) were now able to explain the meaning of the solution $n = 7$ saying, for example, “For the rules to be equal, *n* has to equal 7” or “That at seven, um, Claudia’s rule will equal to Adam’s rule.” Five other students simply stated that *n* was equal to 7 and four gave wrong or uninterpretable explanations.

Students’ Difficulties

Although each student’s interview was unique, most of their difficulties, occurring in isolation or combined, could be classified in two main groups: those related to the representation of the unknown quantity (six students) and those emerging from operating on multiple values of the unknown quantity (nine students), with three students presenting both kinds of difficulties.

DISCUSSION

Students’ performance at the end of the three-year early algebra intervention was generally encouraging. In the written assessment, more than two thirds of the fifth graders in the study could produce the equation representing the problem and nearly half of them solved the equation, which included a variable on both sides of the equals sign. In the follow up interview, in interaction with the interviewer, the percentage of students finding the correct solution to the equation increased to 82%. This is no trivial achievement if we consider previous studies on middle and high school students’ difficulties with algebra (e.g., Filloy & Rojano, 1989). In the interview, only four students ultimately failed to solve the equation, displaying various types of difficulties. Moreover, half of those who solved the equation could explain its meaning in relation to the verbal problem they were given.

While previous research has shown that even high school students do not realize that the left and right sides of the equation are no longer equivalent when an incorrect value is used in an equation, seven of our fifth graders explicitly stated that using a value other than the solution to the equation would destroy the equality.

A few students were not initially successful in writing expressions containing an unknown. Some of them attempted to deal with this by trying to instantiate the problem with specific numbers. This indicates that they are still in the process of accepting that one can work with an unknown quantity using the same arithmetical rules that one applies to numbers. However, when the problem was structured with the unknown in place, some of these same students were able to solve the equation.

Previous research has highlighted the difficulties that older students, including adolescents, have in dealing with equations. Moreover, the transition from the semantics of the problem to the syntactic rules of algebra is a rather challenging task that our project tried to address, even though much more research is needed to understand how students can be guided towards learning, using, and understanding the syntactic rules of algebra. Despite these difficulties, our preliminary data show that young students can use algebraic notation to represent verbal problems and attain some degree of success in manipulating symbols to solve equations with a single variable represented on both sides of the equals sign. Their difficulties may be related to the arithmetic curriculum students are exposed to in elementary school, not to cognitive-development constraints. Our data suggest that a functional approach to early algebra and opportunities to discuss the logic implicit in the syntactic rules of algebra can help elementary school students to learn basic algebraic representations and procedures. As we showed in a follow-up study of students in this project (Schliemann, Carraher, & Brizuela, 2012), participating students were better prepared to expand their understanding of algebra in middle and high school than their control group peers.

We hope our work will provide resources for the implementation of algebra in elementary schools, as recommended by the National Council of Teachers of Mathematics (2000), and will contribute to a balanced view of what young children can do regarding the understanding of functions and solution of equations.

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