ALGEBRA IN ELEMENTARY SCHOOL

Analúcia D. Schliemann*, David W. Carraher**,
Bárbara M. Brizuela*

L'ALGÈBRE À L'ÉCOLE ÉLÉMENTAIRE

Résumé – Ce texte porte sur les résultats d'une intervention scolaire de trois ans où des élèves de huit à onze ans (grades 3 à 5) ont suivi des leçons hebdomadaires d'algèbre « précoce ». Nous rendons également compte de l'influence du projet sur l'apprentissage en algèbre des élèves aux grades 7 et 8 par rapport à un groupe témoin. Les élèves qui avaient participé à l'intervention ont dépassé leurs pairs à tous les grades et mieux profité de l'enseignement d'algèbre aux grades 7 et 8.

Mots clés : algèbre, algèbre précoce, variables, fonctions, tables, lignes de nombre, graphe cartésien, équations.

TITLE OF THE ARTICLE IN SPANISH (STYLE RESUME TITRE)

Resumen – Spanish text of the abstract (style resume texte)

Palabras-claves: in Spanish with no capital letters, separated by commas.

ABSTRACT

We report on a three-year classroom intervention study in which eight to eleven year-old children (third to fifth graders) participated in weekly early algebra lessons. We also report on the project's influence on students' subsequent learning of algebra in seventh and eighth grades as compared to a control group. The treatment students outperformed their peers in all grades and better profited from algebra instruction in grades 7 and 8.

Key words: algebra, early algebra, variables, functions, function tables, number lines, Cartesian coordinate graphs, equations.

.

---

* Tufts University, Medford, MA, Etats-Unis – aschliem@tufts.edu, barbara.brizuela@tufts.edu

** TERC, Cambridge, MA, Etats-Unis – david_carraher@mac.com

INTRODUCTION

The role of algebra in elementary school mathematics is the subject of considerable debate in the United States. Traditionally, algebra makes its debut in Algebra I, when students are about 14 years of age. Until then, the school curriculum focuses almost exclusively on arithmetic, with eventual "pre-algebra courses" in the middle grades to ease the transition from arithmetic to algebra.

We claim that algebra has an important role to play in the early mathematics curriculum well before students actually solve equations. We would propose that some of students' difficulties with algebra stem from arithmetic being introduced as if it were unrelated to algebra. For instance, students' asymmetric and intransitive interpretation of the equal sign as signifying "makes" or "yields" reflects how it is introduced in elementary school mathematics.

It may be possible to substantially reduce students' difficulties with algebra by introducing them early on to the ideas and tools of algebra (Booth 1988 and Schliemann, Carraher, & Brizuela 2007). Early algebra could lend coherence and depth to school mathematics and avoid high levels of failure in mathematics, especially among children from minority and low-income populations (Kaput 1998). However, in the United States, even though the National Council of Teachers of Mathematics (NCTM 2000) has endorsed the view that arithmetic and algebra need to go hand and hand in the early grades, this vision is almost never realized in actual classrooms.

The slow pace of change is understandable. Many people believe that the well-documented difficulties with algebra flow from developmental constraints and the inherent abstractness of algebra. To be fair, there is relatively little research, much less research-based curricula, bearing on such an endeavor. However, the situation is beginning to improve (see Kaput, Carraher, & Blanton 2008 and a review by Carraher & Schliemann 2007).

Algebra in the early grades is both an educational endeavor and a field of research. It takes on a variety of forms but it is generally less about starting high school algebra in the early years than helping young students understand more deeply the current topics of the early mathematics curriculum, express mathematical generalizations, and develop symbolic representational systems that will become increasingly useful and generative. Because the form algebra takes in elementary school must depend on what young learners are capable of, we need to capture and nurture expressions of algebraic reasoning that may develop before students have fully mastered algebraic notation.

It is useful to distinguish three ideal types (in Durkheim's sense) of approaches to early algebra, even though proposals will typically have features of more than one approach.

The *generalized arithmetic approach* to early algebra focuses on mathematical structures and properties of number systems such as the commutative, associative, inverse, identity, and distributive properties of addition and multiplication –known collectively as the field axioms. It is exemplified by the work of Carpenter, Franke and Levi (2003) and of Bastable and Schifter (2008), who encourage students to explore basic properties of number systems through individual interviews or classroom discussions about the meaning of number sentences. They explore implicit algebraic thinking and awareness of how the operations are related in children's strategies to solve arithmetic problems.

A *modeling quantitative relations approach* to early algebra (e.g., Davydov 1991 and Dougherty 2008) focuses on counts (e.g., the number of chairs in a classroom, the number of boys and girls), measures of quantities (e.g., a distance in meters, elapsed time in seconds), and physical magnitudes (e.g., unmarked line segments) as opposed to pure numbers. Here, children's mathematical understanding is grounded from the start in extra-mathematical situations. Modeling is thus a means by which mathematical knowledge is acquired. Students are encouraged to use line segment diagrams and algebraic notation for comparing magnitudes and representing relations among givens and unknown values.

A *functions-based approach* to early algebra (e.g., Blanton 2008, Carraher, Schliemann, & Schwartz 2008, Moss & Beatty 2006 and Schliemann, Carraher, & Brizuela 2007) relies on the importance accorded to sets of values and to ordered pairs from a domain and target. In principle, a functions-based approach is compatible with a generalized arithmetic approach, provided that literals (letters such as x, y, z) are treated as standing for all possible values in the respective domain or target.

In our functional approach to early algebra we incorporate some features of each of the three approaches. For example, we have attempted to generalize key arithmetical ideas through functions. A basic notion is that the operations of arithmetic can be interpreted as functions. Multiplication by 7, for instance, is construed as a relation of a set of inputs (x) to a set of outputs such that each input value corresponds to a single output value. As students become familiar with fractions, negative numbers and so forth, they can extend their understanding of multiplication to these domains and are just a short step from the expression of a linear function such as $7x + b$. Within our approach, equations can be represented as the constraining of two functions to be equal. For example, $5 + x = 8$ can be treated as the setting equal of the functions, $f(x) = 5 + x$ and $g(x) = 8$. There is only one solution that makes the equation $5 + x = 8$ true. All other values of x lead to syntactically correct (but false) statements.

Another fundamental feature of our approach to early algebra is that conventional algebraic representations (e.g., functions tables, Cartesian coordinate graphs (hereafter: graphs), and algebraic notation) are treated as alternative representations of the same abstract mathematical objects. Students are not going to reinvent these essential tools for expressing functional relationships on their own. However, their invented and idiosyncratic representations can pave the way for use of conventional means.

Below we describe one of our longitudinal studies where third to fifth grade students (8 to 11 year-olds) were introduced to elementary algebra within a functions-based approach. In previous descriptions of discussions and classroom based work during our early algebra lessons and interviews, we showed that young students are capable of algebraic reasoning using mixtures of self-invented and conventional symbols (see Brizuela & Earnest 2008, Brizuela & Schliemann 2004, Carraher, Schliemann, & Brizuela 2005, Carraher, Schliemann, Brizuela, & Earnest 2006, Carraher, Schliemann, & Schwartz 2008, Carraher, Martinez, & Schliemann 2008, Peled & Carraher 2008 and Schliemann, Carraher, & Brizuela 2007). Here we compare results of the three-year intervention on the experimental and control students at the end of fifth grade and, later, in seventh and eighth grade. We examine their performance on written assessments to determine whether early algebra instruction prepares the way for algebra proper.

OUR THIRD TO FIFTH GRADE INTERVENTION AND ITS IMPACT

1. Participants

From 2003-2005, we worked with an initial cohort of 26 students who attended an urban public school in the Boston area implementing two 60-minute lessons per week in third and fourth grades and one weekly 90-minute lesson in fifth grade (50 lessons in third grade, 36 lessons in fourth grade and 18 lessons in fifth grade). Each lesson was followed by homework designed by the research team and 30 minute homework review sessions led by each classroom's teacher. During 2004 and 2005, the school's regular teachers taught the third and fourth grade lessons to a new cohort of 24 students. The control group in this longitudinal study consisted of 30 students in the cohort immediately preceding the first cohort, who had been taught by the same teachers from third to fifth grade each previous year. In comparison to the control group, students in the experimental group had one extra hour of mathematics instruction per week, throughout the project.

We lost contact with many of the experimental students when they completed elementary school; this reduced the experimental group for the follow-up study to 20 students. Each year we administered to them and to a control group of peers a written assessment. When the experimental group students had concluded seventh and eighth grades (at 12 to 14 years of age) we offered a one-week algebra Summer Camp that was attended by six of them and by 19 control students. We collected assessment data for these students before and after the Summer Camp and from the other 14 experimental group students at their homes.

We will next describe the implementation of several key lessons in the third to fifth grade intervention as well as in the Summer Camp. Then we will compare experimental and control group written assessments at: (a) the end of fifth grade; (b) the end of seventh and eighth grade; and (c) before and after participation in more advanced algebra lessons in the Summer Camp.

2. Procedure

Our lessons (see http://www.earlyalgebra.org) in third, fourth, and fifth grade focused on functions and their representation through natural language, function tables, graphs, and algebraic notation. Lessons focused upon rich problem contexts and situations involving relations between physical quantities. Students gradually learned to use letters to stand for unknown amounts and to establish correspondences across various representations of a given problem. Algebraic notation was introduced first to help them express what they already knew from reasoning about the problems without algebraic notation. Over time the notation served to help them derive new information and insights. In addition, we encouraged students to derive conclusions directly from mathematical representations such as graphs or algebraic expressions and, ultimately, to solve equations using algebraic methods.

Understanding and Representing Variables and Functions.

In early third grade, lessons aimed at helping students to note and articulate relations among variables. In one task, *The Candy Boxes Problem*, the instructor holds a box of candies in each hand and tells the students that the box on the left is John's and the box in the right is Mary's; each of the two boxes is said to have exactly the same number of candies inside; John also has one extra candy and Mary had three additional candies resting atop their box. Students discussed how many candies were in the boxes and used words, numbers, or drawings to describe the situation on paper. We were interested in whether the children would fix the amounts or consider multiple possible amounts (in accordance with the concept of variable).

Students initially attempted to guess the number of candies in the boxes (shaking the boxes to determine how many candies were in the boxes was a frequent strategy). However a few students left the value undefined: they drew the boxes as opaque, drew a question mark on each of the boxes, or expressed that they were reluctant to assign a specific value because they did not know what it was. The instructor took their reluctance as an opportunity to introduce a letter to stand for "whatever the number of candies happened to be inside the box". Students generally found this idea acceptable. However, it was not immediately evident that one could operate on the *letter* to express John's total and Mary's totals as a mathematical expression or as a function of N. Students did spontaneously state that John had "N and one" and that Mary had "N and three" candies. But it generally required some discussion for them to suggest and convince their classmates that these amounts could be written as "$N + 1$" and "$N + 3$". Furthermore, discussion was necessary for students to realize that the letter N must stand for the same number in each instance. Whether N stood for all possible numbers in the domain or merely the number that happened to be true at the moment required discussion. As students proposed their 'guesses' to the pair of possible corresponding values for John and Mary's totals (notably, a guess is a single value), the instructor recorded the values, one row for each guess, in a table with headers for John's total and Mary's total. Although in practical terms only one solution could in fact hold for the case at hand, the problem might be represented by a(n infinite) number of ordered pairs. By shifting the concern from actual values ("How many candies are in the boxes?") to possible values ("How many candies could be in the boxes, according to the problem?") the discussion moved away from particular numbers to sets of numbers.

In subsequent lessons, students explored relationships between variables through number lines, function tables, and algebraic notation. Guess-My-Rule games, verbal problems involving variables and unknowns, and the algebraic representation of functions underlying patterns were contexts for discussions about relations between variables.

Number Lines, Graphs, Cartesian Space, and Linear Functions.

In third grade, students learned to represent additive operations on number lines by displacing themselves along lines drawn on the classroom floor or by drawing displacements on lines on paper. We also introduced them to the 'N-number line'—a variation on a standard number line in which positions were identified as $N - 2$, $N - 1$, N, $N + 1$, $N + 2$ etc. Students expressed additive operations on the N-number line to solve addition and subtraction problems where the starting point had no determined value.

At the end of third grade we asked them to keep track, on two parallel number lines drawn on the classroom floor, of a pair of values corresponding to quantities in the statement "For each hour of work you get $2". Students were to stand on two points (one on each number line) corresponding to the elements of an ordered pair. This proved easy for close values but became increasingly difficult as the distance between the points increased.

We then suggested that one of the number lines (amount paid) be rotated to lie perpendicular to the other (hours worked). This transformed the number lines into the axes of a coordinate space on the classroom floor, where a single point could now represent two values simultaneously. This Cartesian representation required the additional assumption regarding projection lines, perpendicular to each axis, all the positions along which share the values of the point on the corresponding axis. With some guidance, students plotted themselves at points in the plane embodying a graph representing the relationship between the variables. In subsequent lessons, they interpreted graphs and built function tables and graphs to represent relations of earnings per hour, candy bars per person, and distance per time.

By the end of fourth grade students worked with problems involving the comparison of two linear functions such as: "Mike and Robin each have some money. Mike has $8 in his hand and the rest of his money is in his wallet. Robin has, altogether, exactly three times as much money as Mike has in his wallet. How much money could there be in Mike's wallet? Who has more money?" At first, students were likely to assume that one boy had more money than the other. By filling out a table of possible values for the amounts in the wallet and the total amounts of money of each protagonist, they came to realize that the answer depended on the amount of money in the wallet. They plotted and compared the graphs of Mike's and Robin's amounts depending on the amount in the wallet, expressing Mike's amount as $N + 8$ and Robin's as $3N$. They further realized that those amounts would be equal only if $N = 4$, that is, if the wallet contained four dollars. This lesson and other fourth grade lessons were meant to convey the idea that equations are special cases where two functions produce the same value and to prepare the students to work, in fifth grade, with equations as algebraic objects to be directly operated upon.

Representing Problems as Equations and Solving Them.

Fifth grade lessons focused on algebraic notation for representing word problems, leading to linear equations with a single variable or with variables on both sides of the equals sign. Students used a template for systematically solving equations by proceeding in a downward fashion that allowed comparisons of each equation to the one that preceded and followed it and by explicitly registering the operations performed on each side of the equal sign. To highlight that new equations, obtained through equal transformations on both sides of the equality, all have the same solution, we implemented *unsolving-the-equation* tasks where one begins with a solution, e.g., $x = 7$, and generates one or more equations that will be true if the solution is true.

The Summer Camp Lessons

The Algebra Summer Camp was offered over nine consecutive week days to experimental and control group students at the end of seventh and eighth grades, two or three years after the early algebra intervention. The camp lessons focused on linear, quadratic, and exponential functions, notations for functions (e.g., $y = 3x$ versus $f(x) = 3x$), the y-intercept, composition of functions, translation and rotation, slope, and the graphical representation of functions in the four quadrants. As in the earlier intervention, each lesson involved classroom discussion and small group work and focused on a single multi-faceted problem, with emphasis on relationships across multiple representations.

Assessments

At the end of fifth grade, students in the experimental and control groups were given a written assessment on problems designed by the research team and questions from standard evaluations of mathematical learning such as the *National Assessment of Educational*

Progress (NAEP) and the *Massachusetts Comprehensive Assessment System* (MCAS) for fourth, sixth, and eighth grades.

At the end of each of three school years following the intervention, students were given written assessments with questions related to the early algebra intervention and more advanced problems, typical of the middle and in high school algebra curriculum in the Boston area. These problems were designed by the research team or taken from the NAEP, MCAS, and TIMSS (*Trends in International Mathematics and Science Study*) assessments for middle and high school.

RESULTS

1. Fifth Grade Results

Of the 50 items in the fifth grade written assessment (end of the intervention), 28 were related to the early algebra intervention lessons and 22 were not. Answers were classified as correct or incorrect.

Figure 1 shows the students' average percent of correct responses by groups at the end of the intervention (fifth grade) for 28 items related and 22 not related to the intervention.

On items related to the intervention, the experimental group performed significantly better than the control group. On items not related to the intervention, the two groups performed equally well.

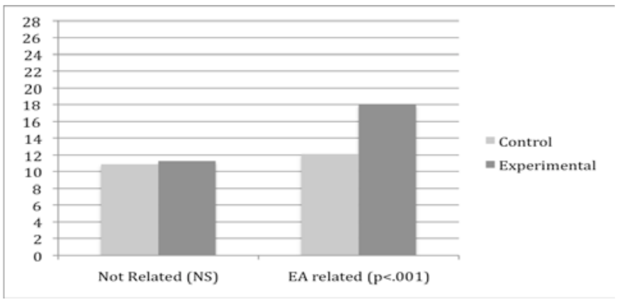


Figure 1 – Average number of correct answers

2. Seventh and Eighth Grade Results

The seventh and eighth grade assessments included 20 items designed to evaluate more advanced understandings of the middle and high school algebra curriculum, which were addressed in our Summer Camp lessons, as opposed to the content of the third to fifth grade intervention.

Fifteen items concerned *Equations*: solving linear equations; solving verbal problems by representing them as equations, solving the equations, and interpreting its results; and determining the equations for functions (including quadratic functions) depicted in patterns or data tables.

Five items involved *Graphs:* the graphical representation of non-linear functions in the four quadrants of the Cartesian space, a topic that was not part of the third to fifth grade intervention. Students were asked to determine which graphs of non-linear functions corresponded to equations or to function tables.

Figure 2 compares the seventh and eighth grade control ($n = 19$) and experimental ($n = 20$) students for *Equations* and *Graphs* items, two years (seventh graders) and three years (eighth graders) after the end of the intervention. Data for the control group and for six experimental group students were collected before the start of the 2009 Summer Camp. Data for the remaining 14 experimental group students were those collected in their homes, since they did not attend the Summer Camp.

In this evaluation of the long term impact of the early algebra intervention, the experimental group performed better than the control group in both types of problems; the difference between the two groups was significant for *Equations*, but not for items on *Graphs*.

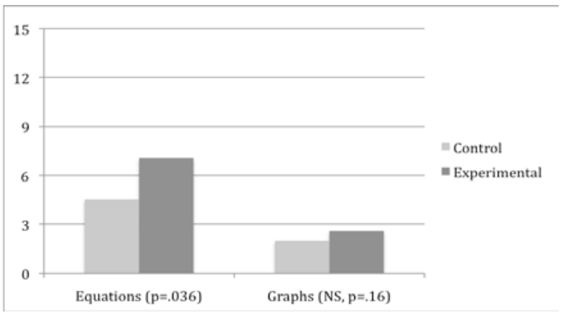


Figure 2 – Average number of correct answers by groups on 15 Equation items and 5 Graphs items

A second analysis (Figures 3 and 4) examined how the experimental and control group students benefitted from the Summer Camp lessons. This analysis included the six experimental and 16 control group students who participated in Summer Camp 2009, taking pre- and post-assessments.

The experimental group performed better than the control group in both sets of assessment items, before and after Summer Camp.

Moreover, the difference between the two groups increased after Sumer Camp participation. Differences were significant before and after Camp for *Equations*. For *Graphs,* the difference between groups was not significant before, but became significant after Camp. These results support the claim that students in the experimental group were better able than their control group peers to benefit from the camp lessons.

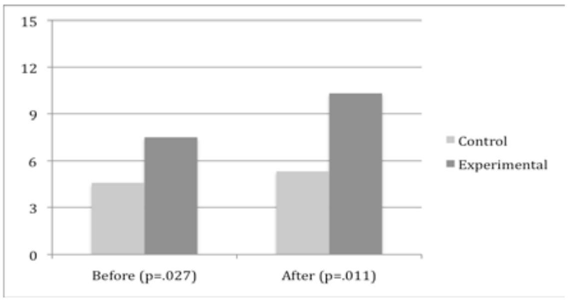


Figure 3 – Average number of correct answers by groups on 15 Equation items, before and after Summer Camp

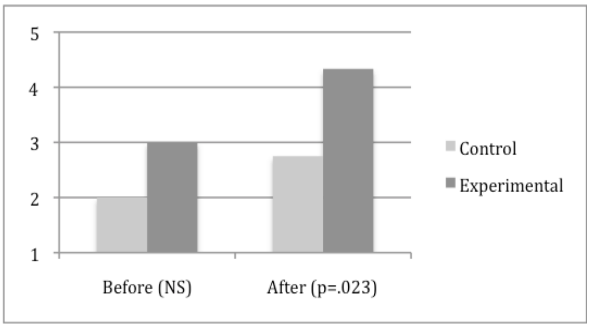


Figure 4 – Average number of correct answers by groups on 5 Graph items, before and after Summer Camp

3. Examples of Assessment Problems

The following problems, two on *Equations* and one on *Graphs*, illustrate the level of difficulty of the written assessments questions.

The first problem was taken from the state-mandated MCAS 10th grade test. It requires using algebraic notation to represent verbal statements, setting up an equation to describe the verbal problem situation, solving the equation, and correctly interpreting the result:

> Liam and Tobet are going to walk in a fund raising event to raise

> money for their school. Liam's mother promised to donate to the school \$4 per mile Liam walks, plus an additional 30. Tobet's father promised to donate to the school \$5 per mile that Tobet walks, plus an additional \$20.
> (a) If Liam walks 15 miles during the event, what is the total amount of money his mother will donate? Show and explain how you got your answer.
> (b) Write an equation that represents y, the total amount of money Liam's mother will donate if Liam walks x miles during the event.
> (c) Write an equation that represents y, the total amount of money Tobet's father will donate if Tobet walks x miles during the event.
> After the event, Liam and Tobet compared their results. Liam had walked the same number of miles as Tobet. Liam's mother had donated the same amount of money as Tobet's father.
> (d) Using your two equations from parts (b) and (c), determine the number of miles Liam and Tobet each walked during the event. Show and explain how you got your answer.
> (e) Using your answer from part (d), determine the total amount of money Liam's mother and Tobet's father each donated. Show or explain how you got your answer.

The average score on this problem for 10th grade students in Massachusetts was 2.80 (out of 4 possible points). Before and after Summer Camp, our experimental group outperformed the control group (1.81 vs. 1.58 points before and 3.83 vs. 2.95 points after). They also outperformed 10th graders attending schools in the same district, two or three years ahead of them.

The second problem asks the student to choose the equation that corresponds to the data presented in a table:

> The table below shows a relationship between values of x and y

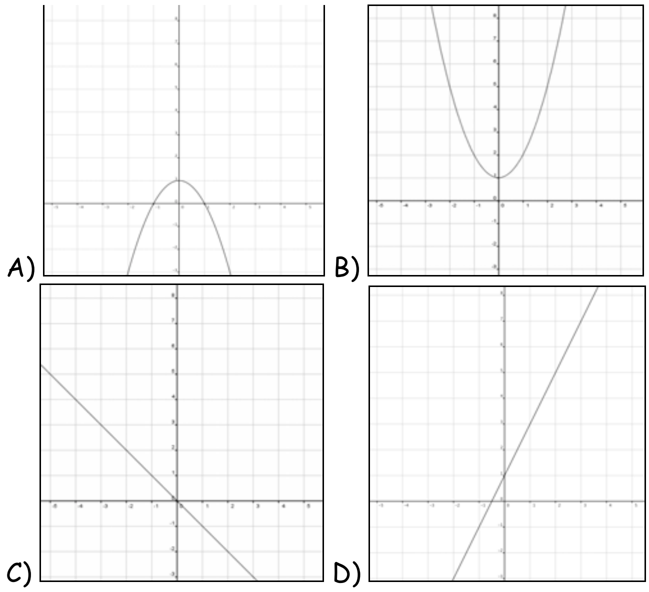
x	1	2	3	4	5
y	3	6	11	18	27

> Which of the following equations describes the relationship between x and y for the values in the table?
> A. $y = 3x$
> B. $y = 5x - 2$
> C. $y = x^2 + 2$
> D. $y = x^3$

For this second *Equations* problem, 75% of students in the experimental group chose the correct answer, while only 37% of control students did so.

The following exemplifies the problems on *Graphs:*

> Which of the following is the graph of the line with equation $y = x^2 + 1$? Circle one.



In this question on *Graphs*, 42% of the control group and 50% of the experimental group students chose the correct answer.

DISCUSSION

We are generally encouraged by the results of the early algebra intervention. At the end of elementary school the treatment students fared better than controls on algebra problems. And the benefits appear to have persisted two to three years later, when they were more successful than their peers in learning to solve more advanced algebra problems.

We cannot know exactly what factors are critical in early algebra learning. As with any educational intervention, there will be questions related to the "shotgun effect". Nonetheless, there is now substantial evidence that young children can handle certain mathematical objects—variables, functions, covariation, algebra notation—that were thought to be more appropriately introduced at adolescence (and even then, with some question about whether all students could handle their cognitive demands). If we had to wager a bet, we would venture that the sort of performance observed here may be routinely reached by young students in coming years, if they are encouraged to reason algebraically. And as mathematics educators continue to document classroom studies of early algebra learning and teaching, the field will hopefully gain more insights into the nature of such learning.

Our data suggest that integrating algebra in the elementary school mathematics curriculum is not only possible, but is also beneficial. The separation of arithmetic and algebra can no longer be justified by a focus on young students' lack of logical and formal cognitive capabilities. More likely, schools have been unable to transform the elementary mathematics curriculum into one that supports, enables, and provides plenty of opportunities for children to explore and experience algebra through generalizations, modeling, functions, and their related representations. Early experiences with algebra pave the way for an understanding of mathematics beyond merely numbers, counting, and arithmetic operations. Proof of this is provided by the way in which our experimental group students were able to profit, while in middle school, from algebraic content and concepts that they had not previously experienced.

Hopefully, research studies such as the one reported here will contribute to changes in the school curricula and practice. But we are sober enough to realize that such changes depend on more than the contributions of researchers. Ultimately, they will happen or not to the extent that we use what we are learning to effect changes in programs of teacher education and professional development.

REFERENCES

BASTABLE V., SCHIFTER D. (2008) Classroom Stories: Examples of Elementary Students Engaged in Early Algebra. In Kaput J. Carraher D. W. Blanton M. (Eds.) *Algebra in the Early Grades* (pp. 165–84). Mahwah, NJ: Erlbaum.

BLANTON M. (2008) *Algebra and the Elementary Classroom: Transforming Thinking, Transforming Practice*. Portsmouth, NH: Heinemann.

BOOTH L. R. (1988). Children's difficulties in beginning algebra. In A.F. Coxford & A.P. Shulte (Eds.) *The Ideas of Algebra, K-12. 1988 Yearbook* (pp.299-306) Reston, VA: The National Council of Teachers of Mathematics.

BRIZUELA B. M., EARNEST, D. (2008) Multiple notational systems and algebraic understandings: The case of the "Best Deal" problem. In Kaput J., Carraher D. W., Blanton M. (Eds.) *Algebra in the Early Grades* (pp. 273–301). Mahwah, NJ: Erlbaum.

BRIZUELA B. M., SCHLIEMANN A. D. (2004) Ten-year-old students solving equations. *For the Learning of Mathematics* 24(2) 33–40.

CARPENTER T. P., FRANKE M. L., LEVI L. (2003) *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School*. Portsmouth, NH: Heinemann.

CARRAHER D. W., MARTINEZ M., SCHLIEMANN A. D. (2008) Early algebra and mathematical generalization. *ZDM – The International Journal on Mathematics Education* 40(1) 3–22.

CARRAHER D. W., SCHLIEMANN A. D. (2007) Early Algebra and Algebraic Reasoning. In Lester F. (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 669–705). Charlotte, NC: Information Age Pub.

CARRAHER D. W., SCHLIEMANN A. D., BRIZUELA B. (2005) Treating operations as functions. In Carraher D., Nemirovsky R. (Eds.) *Monographs of the Journal for Research in Mathematics Education* XIII, CD-Rom Only Issue.

CARRAHER D. W., SCHLIEMANN A. D., SCHWARTZ J. L. (2008) Early algebra is not the same as algebra early. In Kaput J., Carraher D. W., Blanton M. (Eds.) *Algebra in the Early Grades* (pp. 235–272). Mahwah, NJ: Erlbaum.

CARRAHER D. W., SCHLIEMANN A. D., BRIZUELA B. M., EARNEST D. (2006) Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education* 37(2) 87–115.

DAVYDOV V. V. (1991) *Psychological abilities of primary school children in learning mathematics (Vol. 6)*. Reston, VA: National Council of Teachers of Mathematics.

DOUGHERTY B. (2008) Measure Up: A Quantitative View of Algebra. In J. Kaput J., Carraher D.W., Blanton M. (Eds.) *Algebra in the Early Grades* (pp. 389–422). Mahwah, NJ: Erlbaum.

KAPUT J. (1998) Transforming algebra from an engine of inequity to an engine of mathematical power by 'algebrafying' the K-12 Curriculum. In National Council of Teachers of Mathematics and Mathematical Sciences Education Board Center for Science, Mathematics and Engineering Education, National Research Council (Sponsors). *The Nature and Role of Algebra in the K-14 Curriculum* (pp. 25–26). Washington, DC: National Academies Press.

KAPUT J., CARRAHER D. W., BLANTON M. (2008) (Eds.) *Algebra in the Early Grades*. Mahwah, NJ, Erlbaum.

MOSS J., BEATTY R. (2006) Knowledge Building in Mathematics: Supporting Collaborative Learning in Pattern Problems. *International Journal of Computer-Supported Collaborative Learning* 1(4) 441–65.

National Council for Teachers of Mathematics (2000) *Principles and standards for school mathematics*. Reston, VA: NCTM.

PELED I., CARRAHER D. W. (2008) Signed numbers and algebraic thinking. In Kaput J., Carraher D.W., Blanton M. (Eds.) *Algebra in the Early Grades* (pp. 303-328) Mahwah, NJ: Erlbaum.

SCHLIEMANN A. D., CARRAHER D. W., BRIZUELA B. M. (2007) *Bringing out the Algebraic Character of Arithmetic: From Children's Ideas to Classroom Practice*. Hillsdale, NJ: Erlbaum.