Chapter 10

Early Algebra Is Not the Same As Algebra Early

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Note[1]

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Early Algebra Is Not The Same As Algebra Early

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Early algebra is not algebra, just earlier. It is a novel approach, or family of approaches, to interpreting and implementing existing topics of early mathematics. Teaching early algebra is not the same as teaching algebra: teachers help students reflect deeply on ordinary topics from early mathematics, to express generalizations and to use symbolic representations that become objects of further analysis and inference. Learning early algebra involves a conceptual shift from particular instances to sets of cases and their relations. Although this shift places significant demands on learners (and on teachers as catalysts of conceptual growth), it is worth the effort. Here we present some of this evidence from our research with children between the ages of 8 and 10 years[1].

Early Algebra ≢ Algebra Early

Many mathematics educators recognize that algebra has a place in the early grades. But they can also identify with Bertrand Russell's remarks:

> "The beginnings of Algebra I found far more difficult [than Euclid's geometry], perhaps as a result of bad teaching. I was made to learn by heart: 'The square of the sum of two numbers is equal to the sum of their squares increased by twice their product'. I had not the vaguest idea what this meant, and when I could not remember the

> words, my tutor threw the book at my head, which did not stimulate my intellect in any way." (Russell 1967)

To move algebra-as-most-of-us-were-taught-it to elementary school is a recipe for disaster. If algebra is meaningless at adolescence, why should it be meaningful several years earlier? Why are increasing numbers of today's mathematics educators embracing early algebra? What guarantees that early algebra will not turn into lumps in the gravy, hostile bacteria in inflamed tissue, excess luggage for our already overburdened syllabi? What is early algebra, if it is not the algebra most of us were taught?

Early algebra differs from algebra as commonly encountered in high school and beyond. It builds heavily on background contexts of problems. It only gradually introduces formal notation. And it is tightly interwoven with topics from the early mathematics curriculum.

1. <u>Early algebra builds on background contexts of problems</u>. The idea that rich problem contexts can support the introduction of algebra may appear to undermine the goal of getting students to use formal notation without having to 'translate' the meaning to mundane contexts. Why immerse students in nuanced discussions about problem contexts if we want them to think ever more abstractly? The justification for building on rich problem contexts rests on how most young students (and many adults) learn. They do not draw conclusions solely through logic and syntactical rules. Instead they use a mix of intuition, beliefs and presumed facts coupled with principled reasoning and argument. We discuss the problem of contexts at length elsewhere (Carraher and Schliemann 2002a; Schliemann and Carraher 2002).We would also draw attention to the insightful analyses of colleagues (Schwartz 1996; Verschaffel, Greer and De Corte 2002; Smith and Thompson 2005). In treating 'situations' as one of the three defining characteristics of mathematical and scientific concepts, Vergnaud has made seminal contributions to the role of contexts in additive and multiplicative

reasoning (Vergnaud 1982, 1994; 1996)[2]. Starting from rich problem contexts and situations one hopes that at some point students will be able to derive conclusions directly from a written system of equations or an x-y graph drawn in a plane. But what assures us that they will ever arrive at this point? This is where the role of the teacher can be decisive.

2. <u>In early algebra formal notation is introduced only gradually.</u> Young students will not reinvent algebra on their own, and without a certain degree of guidance they are unlikely to express a need for a written notation for variables. Algebraic expressions need to be introduced, but introduced judiciously, so as to avoid 'premature formalization' (Piaget 1964). Teachers need to introduce unfamiliar terms, representations and techniques, despite the irony that in the beginning students will not understand such things as they were intended[3]. The initial awkwardness vis-a-vis new representations should gradually dissipate, especially if teachers listen to students' interpretations and provide students with opportunities to expand and adjust their understandings[4].

3. <u>Early algebra tightly interweaves existing topics of early mathematics</u>. It makes little sense to append early algebra to existing syllabi. Algebra resides quietly within the early mathematics curriculum—in word problems, in topics (addition, subtraction, multiplication, division, ratio and proportion, rational numbers, measurement), and in representational systems (number lines and graphs, tables, written arithmetical notation, and explanatory structures). The teacher helps it emerge; that is, she helps bring the algebraic character of elementary mathematics into public view.

In this chapter we will discuss these three distinguishing characteristics of early algebra[5], drawing upon examples from our longitudinal investigations of four classrooms in an ethnically diverse school in the Greater Boston area. From the second half of grade two to the end of grade four, we designed and implemented weekly early algebra activities in the classrooms. Each semester students participated in six to eight activities, each activity lasting for ninety minutes. The activities related to addition, subtraction, multiplication, division, fractions, ratio, proportion, and negative numbers. The project documented how the students worked with variables, functions, positive and negative numbers, algebraic notation, function tables, graphs, and equations in the classroom and in interviews (Carraher, Brizuela and Earnest 2001; Carraher, Schliemann and Brizuela 2001; Schliemann and Carraher 2002; Carraher and Earnest 2003; Schliemann, Carraher, Brizuela and Earnest 2003; Brizuela and Schliemann 2004). To highlight the nature of the progress students can make in early algebra, we will compare the same students' reasoning and problem solving at the beginning of grade three and in the middle of grade four. We will show how mathematics educators can exploit topics and discussions so as to bring out the algebraic character of elementary mathematics.

From Particular To General: The Candy Boxes Problem

To exemplify how young students make initial sense of variables and variation in mathematics, we begin with our first lesson in one of the classes from grade three. The students are eight years old. In one of the classes the instructor, David, the first author, holds a box of candies in each hand. He tells the students that:

- The box in his left hand is John's, and all of John's candies are in that box.
- The box in his right hand is Mary's, and Mary's candies include those in the box as well as three additional candies resting atop the box.
- Each box has exactly the same number of candies inside.

He then invites the students to say what they know about the number of candies John and Maria have. At a certain point he passes around one of the boxes so that students can examine it; rubber bands secure it shut and students are asked not to open it.

What Students Focused On

Students invariably shake the box (Figure 1), seeking to appraise its contents. After having held and shaken the box, most make a specific prediction, even though the instructor has not requested that they do so. One student holds a box in each hand and concludes that the boxes hold differing amounts of candy. Several others appraise the weight of the two boxes. Another conjectures that there are no candies at all in the boxes. David explains that he placed tissue in the boxes to muffle the sound and make guessing difficult; the students find this preventive measure amusing. Eric ventures, with a gleam in his eye, that there is a doughnut in one of the boxes—a comment his classmates find delightfully funny, perhaps because it violates 'given facts' David is trying to establish. David insists that he put the exactly same number of candies in each box; he appeals to the students to accept his word. The students appear to accept his claim and to take pleasure in having raised reasonable doubts.

Figure 1: A student shakes one of the candy boxes listening closely to estimate the number of candies.

After 15 minutes of discussion David asked the students to express in writing what they knew about the amounts John and Mary had. If a student balked or stated that she did not know how many candies they had, David encouraged the student to show what she did know and to show how she was thinking about he story. Fifty-six out of 63 children produced drawings. Two distinct foci emerged.

A Single Instance

The first focus consisted in ascribing a particular value to the amounts. Forty of the 63 children (63.4%) focused on a single case. That is, they used drawings, labels, or prose to assign particular numbers to the amounts John and Mary had.

Figure 2 illustrates such a focus on a single case. In Erica's drawing the candies are shown in a top row of a table[6]; the respective owners, John and Mary, are identified through drawings and labels in the next row. Erica's representation considers only one case or instance: the case of six candies in each bag, giving John and Mary 6 and 9 candies, respectively. Erica also expresses the single case through the number sentences, "Six + three=nine" and "●●●●●●+●●●=nine."

Figure 2: Erica's representation of John and Mary's candies.

Under questioning Erica recognizes that there are other possible answers:

> [1] Darrell [*interviewing Erica in class immediately after she produced her drawing*]: Do you think there are any other guesses you can make?
>
> [2] Erica: Yeah.
>
> [3] Darrell: Yeah? What are some other guesses?
>
> [4] Erica: You can put seven, and you can have eight. You can have more.

At the beginning of grade three many students conceptualize the situation as Erica does: given indeterminate amounts, they assign particular values, even though they realize they are hazarding a guess.

An Indeterminate Amount 'Keeps Options Open'

Twenty-three children (36.5%) refrained from assigning values to the amounts of candies in the box. Vilda's annotated drawing (Figure 3 below) illustrates this approach: "I thick Mary's have more den John because Mary's have 3 more candy den John. If you taek 3 She vill have the Same of John (sic)."

Figure 3: Vilda leaves indeterminate the number of candies in the boxes

Although Vilda does not speculate on the amounts John and Mary have, she states that the children would have the same amounts if three were taken away from Mary. Several children simply stated in writing that Mary had three more candies than John or that John had three less than Mary, much as Vilda did. We considered these responses to be similar insofar as they leave options open.

Occasionally a student will attempt to explicitly represent the indeterminate amounts: In another classroom, Felipe (Figure 4) chose to highlight with question marks the fact that the amounts in the boxes are unknown. When the teacher asked him to provide more detail, he ventured a guess: "My guess for the candies is that their (sic) are 8." Felipe's thinking nonetheless emphasizes the indeterminate nature of the amounts. This is not quite the same as conceptualizing the amounts as variable quantities, but a representation of an indeterminate amount is a placeholder for the eventual introduction of a variable.

Figure 4: Felipe's question marks explicitly represent the unknown amounts in the candy boxes. The vertical lines drawn in the middle of each box are the rubber bands that hold the boxes shut.

It might appear to be of no significance that a student expresses the indeterminate nature of amounts. After all, the amounts were presented by the instructor as indeterminate: there was 'some amount' in each box[7]. However, students show restraint in deciding to leave the possibilities open. And, as we shall see, the students who insist upon leaving the values indeterminate provide an important opportunity for the teacher to introduce new notational forms that will prove useful not only in this setting but also in many other future discussions.

Intervening to Shift the Focus and Broaden the Discussion

So far the class appears to be discussing a particular story about two children and their candies. Yet it is possible to conceive of the candy boxes as a *collection of many possible stories*. The teacher hopes to gradually move the focus of discussion towards algebra by capitalizing on this shift in thinking.

Tables Draw Attention To Multiple Possibilities

Back in the original class, David proceeds to summarize certain features of students' representations in a data table with columns for students' names and the amounts they suggested for John and Mary. Later he adds an additional column for keeping track of the differences between John and Mary's amounts (see Figure 5).

Figure 5: A table of possible outcomes. Students' names are on the left.

Students who already made predictions in their representations merely need to re-state the amounts; some changed their responses as a result of having a new 'hunch'. Students who had not specified particular amounts in their drawings are now asked to suggest possible values.

It might seem that a 'prediction table' would merely reinforce the students' natural tendencies to focus on a single case. Surely this is not the aim of a lesson designed to elicit algebraic thinking among students, where the emphasis moves towards generalization. However, listing the individual cases (students' predictions) serves to highlight the *multiple* possibilities. Furthermore, issues of logical consistency come to the fore under such circumstances. Both of these characteristics are desirable in bringing out the algebraic character of the story.

By listening to other students' conjectures, all the students have the opportunity to think more deeply about the problem. For example Dylan writes '5' in John's column and '5' in Mary's column as well. This leads David to ask whether John and Mary could have the same total amounts. Dylan has been thinking only of the amounts inside the boxes, without taking into account the three extra candies

Mary had on top of the box. When Dylan realizes that column three should list the 'total amount Mary has' he amends his answer to 8.

Students occasionally give answers that violate the given premises. For example, Chris suggests that John has 7 and Mary has 13 candies (see Figure 5). Several students notice the inconsistency and eagerly explain why this cannot be the case. But because Chris himself is not yet convinced, David leaves Chris' predictions in the table for the time being. When the table is almost completed, David shifts attention to the differences in amounts of John and Mary. Several students insist that the differences have to be three. David tries to assume the role of a devil's advocate (*Can't John have 7?* [yes] *Can't Mary have 13?* [yes]. *So, what's the problem?*). Students argue that even though they do not know what Mary and John have, some answers (ordered pairs) are not 'right'. Soon all students appear to agree that, although John and Mary could in principle have any amount, once an amount is assigned to one of them, the other amount can no longer be 'anything'; that is, it is no longer free to vary because the variable has been constrained to a single solution.

So it is not the case that 'anything goes' when values are indeterminate. We don't know how many candies John has. We don't know how many candies Mary has. Yet it cannot be true, for example, that John has 6 candies and Mary has 7 candies. By drawing attention to this, issues of a more general nature begin to emerge.

Letters Can Name Indeterminate Amounts, Setting The Stage For Variables

Students who draw attention to the indeterminacy of the amounts offer an excellent opportunity for instructional intervention. In another classroom Kevin writes that John has three candies fewer than Mary without assigning values to either amount. Likewise, he becomes silent when David asks him to state possible values for the amounts of candy. When it is Matthew's turn to predict a possible outcome, he also balks. David recalls that Matthew, like Kevin, had preferred not to make a prediction in his drawing. So he turns to Matthew, hoping to introduce the algebraic convention that a letter can represent an indeterminate or a variable amount:

[5] Matthew: Actually well I... I well... I think without the three maybe it's... [*having a change of heart*] Yea, I pretty much don't wanna make a prediction.

[6] David [*seizing the opportunity to introduce a new idea*]: Okay, but let me offer you an alternative and see if you're willing to do this… What if I tell you, Matthew, that John has N... N pieces of candy. And N can mean any amount. It could mean nothing. It could mean ninety. It could mean seven. Does that sound okay?

[7] Matthew [*cautiously, without a lot of conviction*]: Yea.

[8] David [*writing N on the blackboard*]: All right, so why don't you write down N. [*Addressing the remaining students*:] He's willing to accept that suggestion.

[9] David [*wondering to himself, "What will the students call Mary's amount?"*]: Well, now here's the problem, and this is a difficult problem. Matthew, how many should we say that <u>Mary has</u> if John has N candies, and N can stand for anything?

Several students suggest that Mary's amount also be called N. This is not unreasonable: after all, David had just told them that "N could stand for anything." However, this is going to lead to trouble[8]. David prolongs the discussion a bit more.

[10] Cristian [raising his hand with great energy, suggesting he has just discovered something]: Oooh. Oooh!

[11] David [*suspecting that Cristian is going to say 'N plus three" but hoping to first give the other students more time to think about the problem*]: Hold on Cristian.

[12] David [*again addressing Joseph*]: What do you think we should do? How would we call... how would we call... if N stands for any amount that he happens to have... okay... that John happens to have, then how much would Mary have? You think she'd have...?

[13] Joseph: N.

[14] David: N?

[15] Joseph: Yes.

[16] Student: Yes.

[17] David: Well, if we write N here [*on the blackboard, next to the N assigned to John*] doesn't that suggest that they [John and Mary] have the same amount?

[18] Several students: No.

[19] Briana [*defending the use of N to describe Mary's amount*]: It could mean <u>anything</u>.

[20] Another student: She could have <u>any</u> amount like John.

[21] David: Yea, it could be anything. I know just what you mean. But some people would look at it and say it's the <u>*same*</u> <u>*anything*</u> if you're calling them both N... Is Mary supposed to have more than John or less or the same?

[22] Students: More.

[23] David: How many more?

[24] Student: Three more.

[25] David: Three more, so could... how could we write down 'three more than N' if N is what John has? How could we do that?

[26] Joey [not Joseph]: Three... cause N could stand for nothing.

[27] David: It [N] <u>could</u> stand for nothing, but we're telling you that we're gonna use it to stand for *any* possibility.

[28] Another student: Nothing.

[29] David [*clarifying to that student*]: Okay, it *could* stand for nothing.

[30] Joseph: N plus three.

[31] David: N plus three?

[32] Joseph: N plus three.

[33] David [*amazed*]: Wow! Explain that to us.

[34] Joseph [dazed, as if he had been speaking to himself]: Huh?

[35] David: Go ahead.

[36] Joseph: I thought, ‘cuz she could have three more than John. Write N plus three ‘cuz she could have any amount plus three.

[37] David: So any amount plus three. So why don't you write that down, N plus three.

[38] Anne [*a member of the research team*]: Cristian had his hand up for a long time too. I was wondering how he was thinking about it.

[39] David: Cristian, [do] you want to explain?

[40] Cristian: I was... I was thinking the same thing.

[41] David: Go ahead, explain. You think the same thing as Joseph, N plus three? [*Cristian nods*.] Why don't you explain to us your reasoning and let's see if it's just like Joseph's.

[42] Cristian [*exemplifying the general relation through a particular case*]: ‘cuz if it's any number, like if it's ninety.

[43] David [*encouraging Cristian to continue explaining*]: Yea…

[44] Cristian: You could just, like add three and it'd be ninety-three.

[45] David: Yes. Yea, this is really, really neat. You guys are... are… What...what should we call that... should we call that the 'Joseph and Cristian Rule'?

[46] Students: [*laughter*]

After mulling over the possible confusion engendered by the use of **N** to designate both John's and Mary's amounts, and prodded by the teacher to find a better way [9], Joseph and Cristian reached the conclusion that Mary's amount should be called "N plus three" [32-44] . This in a significant step in the direction of using algebraic notation. Joseph and Cristian are using N not simply as a label[9]. In appending the expression "+3" to the "N", they are effectively operating on N.

Summary Of Findings From The Candy Boxes Problem

By asking the students to make predictions about numbers of candies, we may have encouraged some of them to construe their task as having to guess accurately. However, this same activity served as an opportunity to discuss "impossible answers", such as when a student suggested that child had 8 candies and the other 10 candies. As the prediction table was completed, students could try to describe what features were invariant among the (valid) answers. In a sense, the data table encouraged students to generalize.

The Candy Boxes task is ambiguous, that is, subject to alternative interpretations. It calls to mind a particular empirical state of affairs as well as a set of logical possibilities. The former empirical viewpoint gains prominence when one wonders how many candies are actually in the box. The logical view emerges as attempts to find multiple "solutions" and express this in some general way.

Each viewpoint has its own version of truth or correctness. Empirically, there is only one answer to the issue about the number of candies John and Mary have. By this standard, only students who ascertain the precise numbers of candies in the boxes can be right. However, the logical standard to which algebra aspires treats as valid all answers consistent with the information given, regardless of whether they correspond to the actual case at hand. Jennifer expressed this point of view clearly at the end of the lesson when asked to say who had given correct answers: “Everybody had the right answer… Because everybody[10]… has three more. Always.”

Some readers may regard such ambiguity as merely a source of confusion that should have been minimized. Yet it turns out to have advantages. The story’s ambiguity allows teacher and students to hold a meaningful conversation even though they may have markedly different initial interpretations.

By engaging in such conversations students can begin to appreciate the tension between realistic considerations and theoretical possibilities (Carraher and Schliemann 2002a; Schliemann and Carraher 2002). This tension arises whenever one uses mathematics to model worldly situations (Carraher and Schliemann 2002b). For example, one can ask whether a host of a party will ever run out of refreshments if, starting with a full liter, she distributes half to the first guest, half of what remains to the second guest, and so on (Stern and Mevarech 1996). In the physical world, the drink eventually runs out when a guest receives the last drop (or molecule). In the world of mathematics, the host can serve refreshments without ever running out because the remaining amount, $(1/2)^n$ liters, never reaches zero liters no matter how great *n* becomes. Rather than regard this as a shortcoming of the problem, one can treat it as a useful illustration of how models serve as simplified approximations that ‘break’ under certain conditions.

The results from the Candy Boxes task suggest that young students may be able to shift their focus from individual instances to sets and their interrelations. In this new conceptual framework the mathematical object is no longer the single case or value but rather the relation, that is, the functional relationship between two variables.

But we should be careful not to over interpret promising first steps. The Candy Boxes lesson represents the beginning of a 'long conversation about N' that will extend over several months and years and in a wide variety of contexts. Let us revisit the students one and one-half years later to see how their thinking changed in the ensuing period.

Comparing Functions: The Wallet Problem

The following episodes come from a unit we implemented at the beginning of the second semester of fourth grade in the same four classrooms. This unit was an extension of children's work on functions. Once again we asked them to provide us with look at how they understood a situation that might be construed in a variety of ways. Here the students, now ten year old, and their instructor considered the following situation:

> Mike has $8 in his hand and the rest of his money is in his wallet;
>
> Robin has exactly 3 times as much money as Mike has in his wallet.
>
> What can you say about the amounts of money Mike and Robin have?

In each classroom, we projected the problem with an overhead, asked students to read the problem out loud.   Sometimes while the projector was off, we asked students to recount the story in their own words.

At the outset two opinions typically arose. Some students took the view that Robin has three times as much money as Mike. Others insisted that Robin had only three times the amount in Mike's wallet. After discussing the various interpretations (and re-reading the story out loud several times), the students reach a general consensus around the second interpretation. We then asked the students to provide us with drawings and explanations showing their understanding of the problem, much as we did for the Candy Boxes task. By this time the students were accustomed to such an open-ended request and they quickly went to work making representations.

From Candy Boxes to Wallets: The Evolution of Children's Representations

In the intervening year and one-half the students' thinking has undergone remarkable transformations.

Consider, for example, the case of Lisandra. When asked to represent the Candy Boxes at the beginning of grade three, she essentially drew pictures showing specific amounts of candies (see Figure 6); she also made a statement about the relation between the amounts[11].

Figure 6: Lisandra's representation of the Candy Boxes Problem at the beginning of grade 3.

But when asked to represent the Wallets problem, her drawings take on a very different role. She draws three wallets for Robin to convey the notion that Robin has three times as much as Mike has in his wallet. Even more striking, she has written the letter N on each wallet. Finally, she has expressed Mike's total as '$N + 8 = \square$' and Robin's amount as $N \times 3 = 3N$.'

Figure 7: In the middle of grade 4, Lisandra represents the amounts of money Mike and Robin have. Note the symbolic use of the wallets with N dollars.

Lisandra's progress is not exceptional. In fact most students (74.6% or 47 of 63) made substantial progress between grades 3 and 4. Like Lisandra, 39 of the 60 three students (61.9%) provided general, algebraic representations of the Wallet problem. Here is a breakdown of the algebraic answers:

a) Conventional notation: Sixteen of the children (25.4%) represented Mike's amount as N+8 and Robin's as N*3 or as an equivalent expression such as N+N+N or 3N (in some cases they used w or r, instead of N). [We included Lisandra in this group; her representation also exhibited characteristics associated with (c).]

b) Implicit operations: Fifteen children (23.8%) used algebraic notation but omitted the + sign in their account of Mike's amount; in other words, they simply wrote 'N 8'(or 'N $8.00'). However, in the case of Robin, only two children left the operation implicit writing 'N N N'. The others chose to write N × 3 (nine children), 3N (two children); and N+N+N (two children).

c) Iconic variables: Eight of the children (22.7%) used wallet icons instead of a letter. Some of these used conventional signs for addition and multiplication; others used the implict operations described in (b).

Approximately one student in eight, (8 of 63) produced drawings or tables with multiple possibilities for the amounts (see Figure 8 below). This type of representation highlights variation and co-variation. Sometimes considerably more is conveyed. For example, Erica's computations on the left and margins highlight what varies (the amount in the wallet) and what remains invariant (in Mike's case, the $8.00; in Robin's case the '× 3').

Figure 8: Erica decided to draw a table showing multiple possibilities for the amounts Mike and Robin had. N.B. The lines were provided by her, not given as part of the problem.

Fewer than one in four students (22.2%) represented the amounts through a single possibility or instance. This compares to nearly two-thirds of the students (63.5%) in the lesson given at the beginning of grade three.

Figure 9 shows how students' thinking changed over the one and one-half year period. There was a dramatic shift in focus. At the beginning of grade three, students thought of the Candy Boxes word problem as a story about two children who had either specific or indeterminate amounts of candies. By the middle of grade four, most of the children conceptualized the problem as a story involving multiple possibilities. Many of those (39 children, or 62.2%) used algebraic notation to capture the functional relationships among the variables. We know from many other studies, including our own, that fourth grade students in the United States do not show this sort of shift in thinking without having learned about algebra; they do not invent such things on their own.

Figure 9: The Growth of Students' Thinking Over One and One-half Years.

This shift in conceptualization allowed the students to further deepen their understanding and technical mastery of mathematics.

Intervening To Enrich The Discussion

Algebraic Table Headers Identify Functions, Streamlining Thought

A week later, David reviewed the wallet problem by having the students help fill in a three-column table projected onto an overhead screen at the front of the class. The original column headings were: "In Mike's Wallet", Mike (in wallet and hand)", and "Robin." Because the students had discussed the problem and provided their personal representations in the prior class, David expected this to be a routine task intended merely to refresh their memories before he would turn toward the graphing the functions. However, he noticed that the students repeatedly asked to be reminded about the details of the word problem. [*What was Michael holding in his hand? What did Robin have?*] Once an amount was suggested as the value in the wallet, zero dollars for instance, the students appeared to need to reconstruct in their minds the situation involving the story's two protagonists. To expedite the process, David added algebraic headers above the original headers: 'W', 'W+8', and '3W', corresponding respectively to the independent variable, Mike's function, and Robin's function.

Figure 10: The table discussed by the whole class (via overhead projector). Note the use of algebraic notation for column headings.

These short inscriptions had a noticeable effect on the collective activity of filling in the table. Once the algebraic column headers were inserted into the table, students were able to quickly supply the values for Mike's and Robin's amounts, given a value for the independent variable, W. In explaining their reasoing it became clear that they no longer needed to think through the problem by imagining Mike's holding $8.00 in his hand, with the rest of the money remaining in his wallet; similarly, they didn't have to reconstruct Robin's amount by parsing the story once again. To obtain Mike's total, they simply added 8 to the value of W in column 1. To obtain Robin's total, they simply multiplied the value of W by 3. Thus the algebraic expressions served as more than column labels. Students used them as cognitive mediators for producing output values for the functions without having to concern themselves with the situation-specific meaning underlying the computations. This procedure is considerably more efficient. It is also very different: using it, students can temporarily disregard the story problem, instead focusing on, and operating on, the written symbols.

This shift, away from semantically-driven and toward syntactically-driven problem-solving, does not signal the end of semantics. Those who use mathematics to model worldly situations (engineers, students, applied statisticians, scientists, and just plain folks, as opposed to pure mathematicians and statisticians), cannot consign semantics and background contexts to the trash bin, for they continue to have important roles in mathematics. Nonetheless, the word 'shift' is appropriate here because young students are gaining familiarity with a domain of mathematical thinking where there can be considerable (and meaningful) inference-making that does not ask for immediate translation back to mundane reality (Resnick 1986).

Graphs Highlight Covariation

The students completed the table of values on individual worksheets. David guided them in plotting Mike's total for the cases when the wallet holds $0, $1.00, $2.00, $3.00 and $4.00; they also plot the total values for Robin when Mike's wallet contains $0, $1 and $2. Figure 11 shows a (poor quality) picture of the projected image at the moment the following dialogue starts. The x-axis was used to represent the amount in Mike's wallet. The y-axis was used to represent the total amount.

Figure 11: Mike's and Robin's amounts plotted as a function of W, the amount in Mike's wallet

A student has just plotted the point (2, 6), corresponding to the case where Mike's wallet contains $2.00 and Robin's total is $6.00.

[47] David (*drawing attention to the colinear points that are beginning to map out two broken lines, one for Mike's and the other for Robin's amounts of money as a function of W*): What's happening to these lines? Does anybody notice anything happening? They don't look parallel to me. Yeah, William?

[48] William [*referring to the increments as one proceeds rightward*]: That Mike is going one by one and uh, Robin is going three by three.

[49] David: Yeah. Robin is going three by three. Can you show us where the 'one by one' and 'three by three are', William? Cause people might not understand what you mean by that. Where is the one by one that you see?

[50] William (*pointing to the line representing Mike's amounts*): Like uh, Mike's not, see, he's going one more up.

[51] David: He goes one up. And then next time he goes one more up, like he goes from six, I'm sorry, from eight to nine to ten to eleven and then to twelve. And what's happening with, uhm, Robin?

[52] William: She [Robin] starts at zero. She goes three and then up to six.

[53] David: Ok. She's only going up by three's.

William appears to be trying describing something like the slopes of the two lines according to the size of the increments by which they grow [48-52]. When it is time to plot the point, (4,12), for Robin's function, David asks the class:

[54] David [*ingenuously*]: Wait a minute, but I thought we already used up that point [*The point, (4, 12), was contained on Mike's graph*]. Can I put another one on there? Can I give the same point to Robin that we give to, to Michael?

[55] Student: Yeah.

[56] Erica: Yeah, 'cuz on number four they were even.

[57] David: Oh, they're even. So how do you know that they're even by looking at the graph? How do you see that they're even? They all look different to me. But how do you know that they have the same amount of money?

[58] Erica: Cause on, on number four they're like, in the same place.

[59] David: The same place? Yes, they are in the same place. Ok.

The realization that the two lines cross when "they are even", expressed by Erica [56] and by other students, is an important step towards equations. It is also a clear example of how the students can interpret the graph in terms of the word problem, that is, to attribute semantics of quantity (Schwartz 1996) based on the syntax (Resnick 1982) of the graph.

The children then move on to complete their tables of possible values and the corresponding graphs. This was easily achieved and, at the end of the lesson, in the four classrooms we worked with, 62 of 63 students (98.4%) completed the table successfully. Thereafter the students completed the graphs on their own worksheets, referring to the values they had entered in their function tables: 57 (90.5%) of the students correctly plotted Mike's and Robin's values.

Graphs can clarify tables and vice versa.

As they finished their work, Anne, one of the researchers present in the class asked Jessie to explain his graph.

[60] Anne: Ok. What do you notice about that graph?

[61] Jessie [*Focusing on the intersection of the two graphs*]: That it crosses over here.

[62] Anne: Can you explain why?

[63] Jessie: Because over here, in the table, it's four and it's twelve, twelve, so they're equal. And then over here it's four and then over here, that's why they cross.

[64] Anne: That's why they cross because what?

[65] Jessie: Cause they are equal in the table.

This is a clear example of what some authors (Brizuela and Earnest 2005) refer to as navigating between diverse representational forms or coordinating diverse representations.

[66] Anne: Ok. What happens down here in the graph? Who has more money on this part of the graph?

[67] Jessie: Uhm, Mike.

[68] Anne: How do you know? How does the graph show that?

[69] Jessie: Because, uhm, Robin's down here and then Mike's all the way up there.

[70] Anne: And then what happens after they meet?

[71] Jessie: Robin goes higher.

[72] Anne: So what does that mean?

[73] Jessie: That Robin gets more money.

Prompted by Anne's questions, Jessie also makes use of the convention that, in a graph, "higher means more" and that, in the particular context they are working with it means "more money" [69-73]. Other children provided similar explanations as they were interviewed in class and during the whole class discussion that followed.

Below is another case where a student used the table as a mean for verifying her analysis derived from the graph.

[74] Anne: How much do they have when they each have the same amount?

[75] Lisandra: Twelve.

[76] Anne: Twelve. Then what happens after that?

[77] Lisandra: Uh, they get different.

[78] Anne: What?

[79] Lisandra: They get different, I mean-

[80] Anne: They get different. Who's gonna have more money after?

[81] Lisandra: Uhm, I think...

Here Lisandra flips back to the page showing her table, carefully inspects it, then turns back to the graph and further inspects the lines.

[82] Lisandra: Uhm, Robin [will have more money after they each have the same amount].

[83] Anne: Robin. How does… How do you know?

[84] Lisandra: Cause Robin's all they way up here [*showing the highest point drawn on Robin's line*].

Solving Equations: The Wallet Problem Revisited

Because the Wallet Problem involves the comparison of intersecting functions, it is suitable for delving into equations. Nonetheless it is important to realize that the problem was not originally put forth as an equation. Doing so would have subdued the functional relations we wished to emphasize. This is easily understood by considering two distinct interpretations of equation $w + 8 = 3 \times w$.

<u>A numerical interpretation of the equation</u>. One might construe the equation, $w + 8 = 3 \times w$, as an equality of the left and right terms, each of which stands for a single number (or measure). If it turns out that the number on the left is the same as the number on the right, then the equation is true. If the number on the left is different from the number on the right, the equation is false. We can refer to an unsolvable equation as indeterminate.

<u>A functional interpretation of the equation</u>. There is a strikingly different way of thinking about an equation, namely, as the setting equal of two functions. This is the interpretation of equations we build towards throughout the early algebra instruction. In this framework, $w + 8$ is a function that is free to vary (take on diverse values) within a specified domain[12], say, the non-negative integers; $3 \times w$ is a different function, presumably in the same domain.

Setting the two functions equal can be expressed by the following equation written in standard symbolic notation: $w + 8 = 3 \times w$. What does this mean? And what consequences does setting the functions equal have?

$w + 8 = 3 \times w$ is true only for the case that w = 4. This is the case when the functions "are even" [56], when "they are equal in the table" [65], when they (the graphs) "cross" [63] or are "in the same place" [58], when they "have the same amount" [74], and so on. Otherwise, the equation is not true, the graphs separate, "they get different" [77], etc. In these latter cases an inequality such as $3 \times w > w + 8$ holds [see 80-84].

It is consistent with the present view that the letter *a* in the equation, $5 + a = 7$, represents a variable, not a single value. The equation is true only when $a = 2$. But *a* is still a variable.

Likewise the equation, $b = b + 1$, is a perfectly sensible equation, even though there is no value of the variable, b, for which the equation is true.

This all may appear to be unnecessary mental gymnastics, but holds a number of important implications for mathematics education. For one thing it implies there is no need to treat unknowns and variables as fundamentally different. We prefer to think of an *unknown* as a variable that for some reason or other happens to be constrained to a single value. This is precisely what happens to *w* when $w + 8$ is set equal to $3 \times w$. The equation holds only for certain values of *w*—actually, only one value. It does not transform *w* from a variable into a single number or instance.

In the example that follows, we invited the children to consider the case where Mike and Robin have the same total amounts of money. The children already knew that this corresponded to the case where Mike's wallet had $4.00 in it. Accordingly, they already knew the solution to the equation before they were asked to solve it. So, at best, working with the equation would appear to offer them no more information. However, the students still had much to learn about how to draw inferences in a new representational system, and it is in this spirit that we introduce the next section.

Setting 8+W equal to 3W

After discussing the tables and the graphs, David writes the equation ‘8 + w = 3 W’ on the blackboard and asks one student to represent Mike by holding a 3x5 card on which $8.00 was written and another on which “W” was written to represent the variable amount in Mike’s wallet. Another child, playing Robin, is given three cards to hold; each one has “W” written on it. In ensuing discussion that followed, David sometimes ‘mistakenly’ addressed the two actors by their actual names. This did not appear to be a source of confusion for the students. So in the transcription below we have replaced the children’s real names with Mike’s and Robin’s names to facilitate reading.

[85] David: If these are equal, if the money here in his hands is equal to, altogether, is equal to all the money that she has, do you know how much money is in the wallet? [*Hoping to find a volunteer*] Do we have any Sherlock Holmes here?

[86] Students: No.

[87] David: Jacky?

[88] Jacky: Four.

[89] David: And how do you know?

[90] Jacky [*thinking of Robin’s case*]: Because uh, four times three is 12.

[91] David: Four times three is 12? And also… So three times four is 12 and?

[92] Jacky [*realizing that he it needs to work for Mike also*]: Eight plus four is 12.

[93] David: Eight plus four is 12.

[94] David: So that's the only way that, that they can have the same amount?

[95] Student: Mh-hm.

They continue.

[96] David [*aiming to simplify the equation by eliminating like amounts for each actor*]: … Can I have them spend some money?

[97] Students: Yes.

[98] David: Ok. I want Mike here to spend everything that's in his wallet. What should I do with the amount that he has?

[99] Student: Take away w.

[100] David [*taking the "W" card from 'Mike's' hand*]: Take away the w. You spent it. Thank you.

[101] David: I picked your pocket, Ok? Just for fun. Are they equal now? Do they have the same amount of money?

[102] Students: No.

[103] David: Well, I wanna keep them equal. How can I keep them equal?

[104] Students: Take away Robin's.

It is not immediately clear to the students how much should be taken from Robin so that her amount will be equal to Mike's diminished amount.

[105] Students: Take away all.

[106] David: Take away all three from Robin? (*David takes away her three "wallets"*) You think they're equal now? [Unclear what the students responded.]

[107] Students [laughing]: No… No…

[108] David: He's got eight dollars. She's left with nothing.

[109] Students: Take, take-

[110] Student: Just take two away.

[111] David: Just take two away? I took one [*David means one W, but this is ambiguous*] away from him when they were equal, so what should I do to [Robin]?

[112] David: I'll go back, remember what I did. I took away his wallet, but I wanna do the same thing to her so that they stay equal.

[113] David [*after returning all the cards to 'Mike' and 'Robin', who have 8 & W, W&W&W, respectively*]: They're equal now, right? I told you that they're equal. So I took away this [*showing the W card in Mike's hand*]. What should I do to Robin's money?

[114] Student: You take away two.

[115] David: How much did I take away?

[116] Student: Two.

[117] David: But I took away one W from him, why should I take double from her?

[118] Student: Because she-

[119] Student [*Still trying to make sense, forgetting that, although Robin has more cards, she and Mike are said to have the same amount of money*]: She has more to loose.

As the discussion proceeds they finally agree on what to do to keep Robin's amount equal to Mike's:

[120] David [*capitalizing on the fact that the students already know the solution to the equation*]: Hold on. How much is in [Mike's] w?

[121] David: They're equal, remember? They're equal. So how much do we know is in the wallet?

[122] Nathan: Four.

[123] David: That's right, Nathan. Does everybody agree there's four dollars here?

[124] Students: Yes.

[125] David: Ok, cause that's the only way they're equal. So how many dollars am I taking away from him?

[126] Student: Two.

[127] Students: Four!

[128] David: Four dollars, Ok? I just took four dollars from him. So how many dollars do I have to take from her?

[129] Student [*It is possible there is a momentary confusion of dollars and cards; but this doesn't explain the answer, "two"]:* Two. One. Four

[130] David: I have to take the same amount!

[131] Students: One! One!

[132] David: One dollar?

[133] Students [*finally getting their referent straight*]: One wallet! One w!

[134] David: Ok, one wallet. Did I take the same amount away from each of them?

[135] Students: Yes.

[136] David: Ok.

[137] Anne: How do you know that?

[138] David: Did I? I took how many dollars away from Mike? How many dollars did I take away?

[139] Students: Four.

[140] David: And how many dollars did I take away from her?

[141] Students: Four.

[142] Student: She has the same amount cause I can see it in her hands.

[143] David: … so [Mike] has eight dollars and [Robin] has...

[144] Student: Eight dollars.

[145] David [*There would be nothing to 'solve' if w is removed from the conversation, so he insists upon referring to what is actually written on Robin's remaining two cards*.]: 2w.

[146] David: Can you tell me what w has to be equal to?

[147] Student: Four, cause four plus four is eight.

A New Equation: 100 + W = 3W

David repeats the same process now with a different function. His aim is to put the students in a situation where they do not already know the answer. He writes the equation 100 + w = 3w on the board. He explains that the situation is now completely different. Then hands William a '100 card' and a 'w card'. He hands Nancy three 'w cards'. Now the students are dealing with an equation for which they don't know the solution. After some discussion, the teacher recommends that they take away one 'w card' from each student.

[148] David: Now, they've still got the same amounts cause we took away the same amounts from each of them.

[149] Students: Oh! Oh!

[150] David: Oh. Oh. William, go ahead.

[151] William [realizing he can now infer the amount in each 'w card']: Uh, Nancy has uhm, fifty each in *w*.

[152] David: Really? So how much is Nancy holding altogether?

[153] William: A hundred.

[154] David: And how much is he [William] holding?

[155] William: A hundred.

Jessie's representations

When David asks "did anybody else realize that it was fifty?", Jessie shows (Figure 12) and explains how he solved the problem in writing:

[156] David: Jessie, how did you do it?

[157] Jessie: Three w's stand for 100.

[158] David: I'm sorry, three?

[159] Jessie (showing his drawing): Three w's and I crossed out a w.

[160] David (explaining to the whole class): He crossed out a w. That was like taking away the w. This is a really, really nice way of doing it. He wrote w, w, w, and that was to stand for what Robin has, right? And then you wrote 100 and w, to stand for what Michael has. You took away a w from each of them, and you were left with two w's and 100. And if two w's is a hundred dollars, each w has to be equal to... fifty dollars.

Figure 12: Jessie's representation of the 3w = 100+w and subsequently 3w-w = 100+w-w.

Jessie's written work shows that he understands how to solve an equation that has sprung from the setting equal of two functions. But this is also the case for students who have managed to solve the equations in the form of index cards and statements, a point we shall soon revisit.

Algebra In Early Mathematics

Reprise: Early Algebra Is Not Algebra Early

We noted at the outset that early algebra is not the same as algebra early. Early algebra builds on the background contexts of problems, only gradually introduces formal notation and tightly interweaves existing topics of early mathematics.

Both the Candy Boxes and the Wallet lessons immersed students in particular background contexts for which they attempted to describe the relationships between physical quantities and ultimately to make mathematical generalizations. As the conversations progressed, we gradually introduced formal representations (tables, graphs, and algebraic-symbolic notation)—where possible, as extensions to students' own representations. Algebra served as a thread that weaves through and helps establish tight bonds across diverse topics (arithmetical operations, variables, sets, additive differences, composition of quantities) and representations (tables, diagrams, number lines and graphs, verbal statements, written symbolic notation).

Now let us shift our attention to the kinds of reasoning that Early Algebra calls for. As we shall see, it engages students in a special kind of generalization.

Deduction Cannot Be The Whole Story

> The very idea of a science of mathematics seems to raise an insoluble contradiction. If this science is deductive only in appearance, from where does its perfect rigor come—a rigor that no one would deny? If, on the other hand, all the propositions mathematics puts forth can be derived from each other through formal

logic rules, will mathematics not be reduced to an immense tautology? (Poincaré 1968/1916, p. 31, translated by the authors)

Mathematics is not entirely deductive. Sometimes it involves thinking about unspoken premises. Sometimes it involves conjectures.

Unspoken Premises

Consider the statement arose in the context of the Candy Boxes problem: "it cannot be the case that John has a total of six candies whereas Mary has a total of 13 candies." At first glance this statement would appear to be necessarily true, given the information that Mary has three more candies than John. But this ignores the fact that students need to think about unspoken premises. The students had to disregard, for example, the possibility that the instructor had misled them or made a mistake in filling the boxes. They further had to assume that the number of candies put into the box remained invariant: none fell out or melted, for example.

As students were passing around the candy boxes for inspection in one of the classes, Joey, a student, accidentally dropped one of Mary's loose candies onto the floor where it shattered. David only noticed the broken candy several minutes later; the shards could be easily seen through the candy's transparent protective wrapping. Joey admitted apologetically that he had accidentally dropped the candy. He seemed concerned that David, the teacher, might be upset.

After assuring Joey that this caused no harm, David asked Joey whether this made a difference for the discussion they were having about the amounts of candies. Note: the boxes still had not been opened. Joey reflected for a moment and then said that Mary had more candies than before. This prompted immediate denials by several students. Others sided with Joey. After giving the matter more thought, Joey concluded that the amount had not changed because if he were to put the shards back together again, they would yield the original amount.

It may appear that conservation of matter cleared up Joey's 'confusion'. But the class discussion had been focusing on the number of pieces without regard to differences in size. By this criterion, Mary arguably had more candies after the candy shattered. Or was the real issue the weights of the candies? If so, what does it mean for two weights to be 'equal' if one only measures weight within a certain margin of error?

Whenever mathematics is used make sense out of data[13], decisions need to be made about the premises that will be honored or disregarded. Even when these matters are settled, decisions need to be made about the mathematical tools useful for making sense of the data. Deductive logic cannot settle all of these issues of modeling because matters of usefulness, cost, and fit depend on human judgment. Furthermore, context-specific considerations may constrain a problem's domain and co-domain. Mathematics education cannot avoid these issues. On the contrary, it needs to raise their profile so that students can assess their germaneness to problems at hand.

Conjectural Generalization

> Mathematicians... always strive to generalize the propositions they have obtained, and... the equation we have been using,
>
> $$a + 1 = 1 + a \qquad \text{[Equation A]}$$
>
> serves to establish the following equation,
>
> $$a + b = b + a \qquad \text{[Equation B]}$$
>
> which is demonstrably more general. Mathematics thus proceeds just like the other sciences, namely, from the particular to the general

(Poincaré, op. Cit., p. 42, translated by the authors, equation captions added.)

Poincaré certainly knew that no amount of deduction would justify the leap from Equation A to Equation B. In fact, this is precisely his point: mathematicians look for opportunities to generalize even when not entitled by the laws of logic.

Now let's imagine that Poincaré had begun with a case involving no variables:

$$7 + 1 = 1 + 7 \qquad \text{[Equation C]}$$

We notice that **7 + 1** can be expressed more generally as ***a* +** 1, which can in turn be expressed more generally as ***a* + *b*.** Thus a numerical expression (7+1) can be regarded as a particular instance of a function (a + b). More boldly, *any* arithmetical statement can be regarded as a particular instance of a more general, algebraic statement and expressed as such through the notation of functions ('7+1 = 1+7" suggests 'a+b = b+a). Arithmetic always affords an opportunity for thinking about algebraic relations.

The generalization of interest here consists in treating an instance (e.g. 7 + 1) as a case of something more general (e.g. $a + b$). We will refer to this as *conjectural generalization* to highlight its non-deductive nature and its relevance to the formulation of mathematical conjectures. The scope widens considerably as variables replace particular values. Attention shifts from number operations to functional relationships. The new yet familiar notation belies the profound shift that has taken place. This is precisely the sort of shift we attempted to promote among our students throughout the *Early Algebra, Early Arithmetic Project* (e.g. Carraher, Brizuela and Earnest 2001; Carraher, Schliemann and Brizuela 2001; Schliemann, Carraher and Brizuela 2001)

Because we worked with indeterminate amounts, the tasks can be interpreted at various levels of generality. This is curious. Mathematics is widely acclaimed for its precision, rigor, and clarity. But ambiguity can be an important resource in teaching and learning. Working algebra successfully into the early mathematics curriculum often hinges precisely on the deft exploitation of ambiguity in problems.

Functions Enable The Shift To Algebra

We have noted how functions such as $a + 1$ and $a + b$ bring to light the general, algebraic character of elementary mathematics. We are not the first to note the critical role of functions. Seldon and Seldon (1992) drew attention to the integrative role functions played in the history of modern mathematics in the introduction to an important work about the suitability of functions as an organizing concept in mathematics education (Dubinsky and Harel 1992). And Schwartz and Yerushalmy (1992; Schwartz and Yerushalmy 1995) developed a broad middle and secondary mathematics curriculum centered around functions.

Although there is general agreement that algebra should become part of the elementary school curriculum (Schoenfeld 1995; National Council of Teachers of Mathematics. 2000), there are varying views regarding the most promising approach for integrating algebra into the early mathematics curriculum. Some have proposed generalized arithmetic (Mason 1996); others focus on the representation of quantities and the solution of equations (Bodanskii 1991). Still others have defended pluralism on the grounds that no single approach can do justice to the range and complexity of algebra (Kaput and Blanton 2005).

We would claim that functions deserve special consideration. All issues of generalized arithmetic can easily be subsumed under functions, but the converse is not true. (The Candy Boxes and Wallet problem are two cases in point. So are most issues from geometry.) Functions are at home in pure mathematical endeavors such as number theory but they are equally at home in applied mathematics, science, engineering and cases where modeling and quantitative reasoning are critical. Functions even provide the tools for data analysis and statistics. Pluralism has a certain appeal, and we would be the last to argue for a 'one size fits all' approach to mathematics education. Nonetheless, the topic of functions merits a top spot as a general organizing theme for early mathematics.

It is nothing short of remarkable that the topic of functions is absent from early mathematics curricula. Although the concept of function arrived late in the history of mathematics, we are finding that students can work with and understand functions at surprisingly early ages. We suspect that the concept of the function can unite a wide range of otherwise isolated topics—number operations, fractions, ratio and proportion, formulas, and so forth—just as it served a unifying role in the history of modern mathematics. It seems to us that curriculum developers, teachers and teacher educators have much to gain by becoming acquainted with functions for mathematics education in the early grades. It will likely take many years for this to happen. And it will require a program of research that puts to test new ideas for early mathematics education.

Functions Need To Be Distinguished From Their Representations

Functions are normally introduced in such a limited fashion that a few words are in order about what they are and are not. As a warm-up exercise, consider the concept of number. In daily life, it makes perfect sense to say that the following are numbers: 8, 7, 0, -43, 3/4, 3.14159… and so on. However in mathematics education, equating numbers with their written forms can lead to serious problems such as the mistaken view that 3/4 and 0.75 are different numbers. But the issue goes deeper. Below (Figure 13) are several representations of the same number; not one of them **is** the number.

Figure 13: Various representations of the number 8: drawing, arithmetical expression, Mayan ideograph, Arabic numeral, position on number line, name in French, Braille (raised and highly raised dots that correspond, downward, to 1000_2, i.e., eight in base 2), Chinese ideograph. None is these is the number 8.

A similar predicament arises in the case of functions. Figure 14 shows four distinct means of representing the same function. Admittedly, these representations are not fully interchangeable. Each representation is likely to highlight certain characteristics of the function. The table tends to be a poor representation for conveying the continuity of a function. The graph conveys continuity, but it can be ill-suited for displaying precise values of the function.

Figure 14: Representations of the same function: a graph, a function table, symbolic notation, and verbal statements.

Going one step further, Figure 15 shows various representations of the same equation. This may surprise those readers who think an equation **is** a written symbolic expression[14]. But if we confuse the written form with the equation itself, that is, a setting equal of two functions, we will fail to recognize when students are working with equations in other formats, for example when our students were making drawings (see Figure 7) of the Wallet problem or trying to describe the equation in their own words. This is not a matter of condescending to accept students' 'personal yet inferior answers.' Research mathematicians acknowledge that functions are validly expressed in language, written notation, graphs, and tables and they rely on these symbolic systems for representing functions in their professional work.

Figure 15: Four representations of the same equation: a drawing (top), graphs, verbal statements, and symbolic notation.

The Growth Of Algebraic Understanding

In our approach we highlight the shift from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations among variables. Whereas our main interest continues to lie in student reasoning, we have found ourselves thrust into the additional role of curriculum developers and teacher educators by virtue of the fact that many young students only show a proclivity to algebra when offered conditions that encourage them to make mathematical generalizations and use particular representations (e.g., graphs and algebraic notation) normally introduced much later.

We witnessed a dramatic shift in students' thinking over one and one-half years. At the beginning of grade three, students interpreted a story with indeterminate quantities as a *single tale* about particular people and amounts. By the middle of grade four most students construed this sort of situation as entailing *many possible stories* involving variable quantities in an invariant functional relationship. A good many of the students made use of algebraic notation to convey the variations and invariance across the stories[15].

The decisive changes in their thinking surprised us. Several years ago, when invited to assess the evidence regarding whether young students could 'do algebra', we downplayed the discontinuity between arithmetic (as generally taught in K-8) and algebra (Carraher, Schliemann and Brizuela 2001). Our initial findings showed that young students could make mathematical generalizations and express them in algebraic notation (Carraher, Schliemann and Brizuela 2000; Carraher, Schliemann and Brizuela 2001; Schliemann, Carraher and Brizuela 2001). Our view at that time was that there did not exist the enormous conceptual leap from arithmetic to algebra that other researchers had proposed; otherwise, young students would not have been able to make such progress in so short a period.

We were opposed to the notion of a ‘cognitive gap’ (Collis 1975; Filloy and Rojano 1989; Herscovics and Linchevski 1994) because we were skeptical about the underlying claim that the transition from arithmetical to algebraic thinking was inherently developmental. We had repeatedly seen authors appeal to the concept of ‘developmental readiness’ to argue that it was unreasonable to expect students to learn *topic x* at a given moment. We had witnessed this in Brazil a quarter of a century earlier, when many people, appealing to developmental readiness (or the lack of it), found nothing particularly surprising in the fact that, for every 100 children who entered the first grade of elementary school, only 50 moved ahead to grade two one year later[16]. What we would call the ‘developmental readiness syndrome’ was well captured by Duckworth (1979): ‘Either we’re too early and they can’t learn it or we’re too late and they know it already”. [This syndrome afflicts many adults, including quite a few educational theorists, developmental psychologists and teachers. Piaget (Inhelder and Piaget 1958) once thought that in order for students to master proportional reasoning they needed to have achieved the stage of formal operations, generally thought to arrive around adolescence and, even then, for a minority of students. He later revised his view to apply only to inverse proportion (Piaget 1968; Schliemann and Carraher 1992).]

Our present findings have convinced us that there is indeed a large leap from thinking in terms of particular numbers and instances to thinking about functional relations. But the fact that most students throughout the United States do not make this transition easily, nor early, may well say more about our failure to offer suitable conditions for them to learn algebra as an integral part of elementary mathematics than it does about the limitations of their mental structures.

What Are Suitable Conditions?

We are still beginning to understand the conditions that promote early algebra learning. It has already become apparent, however, that certain representational forms play a major role. The first of these are the representations students themselves bring to bear on problems. We gave various examples in the present chapter of children's drawings, tables, and verbal comments. And we tried to show how they are important as points of departure for the introduction of conventional mathematical representational forms.

Tables play an important role in urging students to register multiple instances of a function; hence the expression, 'function table.' Even when students learn about functions while working with highly engaging instruments such as a pulley, little may be learned unless students take care to transcribe the data to a table (Meira 1998). But filling in a table is of little use in itself. Students need to scan the table locally and globally in search of generalizations that can be used to predict outputs from inputs and extend the table to cases for which data do not yet exist. In this regard it can be very helpful to have students register the data as uncompleted calculations; such expressions facilitate the detection of variation and invariance throughout the table. The ultimate test of whether a table is being employed as a function table is whether or not a student can express in general fashion an underlying rule for an arbitrary entry. This amounts to the recognition that the underlying rule is a "recipe" (i.e., a function) for computing an output for *any* allowable input in the domain of the "recipe" (see, for example, Schwartz and Yerushalmy 1992, 1995).

Once students are comfortable with symbolic notation for functions, symbolic table headers can be employed, allowing students to reason about relationships with a diminished need to verify the meaning of the data in terms of the semantics of the situation. Eventually the syntactical moves will acquire a logic of their own, and the student can temporarily disregard the meaning of the symbols, deriving conclusions from the structure of the written forms and the current rules of inference.

Over the course of time problems can be introduced in the form of written symbolic notation, graphs and tables, for which the students are asked to generate appropriate meaningful situations. Furthermore, students can be asked to envision how transformations within one representational system manifest themselves in another (e.g. if the graph of a function is displaced upwards by 3 units, how does this change the associate real-life story underlying the graph?).

As significant as our students' progress may be, algebra is an vast domain that can allow for continued learning and intellectual growth over many years. There will be new functions and structures to become familiar with. The shift from thinking about instances to functional relations may well resurrect itself at other moments along students' intellectual trajectories. Rasmussen (Rasmussen 2004) found, for example, that in learning about differential equations university students first focus on local differences in values (deltas) along x and y, and slopes at a single point. Only later do they conceive of slope as a derivative, that is, as a function that comprises all the particular instances of slopes all along the graph. As educators increasingly implement programs of early algebra at the elementary level, there will be many opportunities for helping how early mathematical learning evolves over many years and helps set the stage for later learning.

Table of Figures

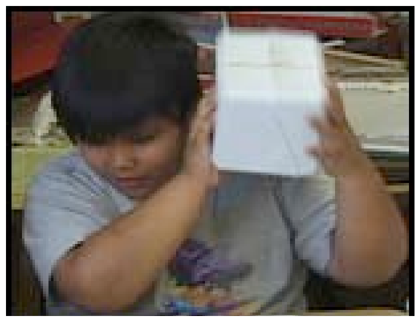


Figure 16: **Student shakes one of the candy boxes listening closely to estimate the number of candies.**

Figure 17: **Erica's representation of John and Mary's candies.**

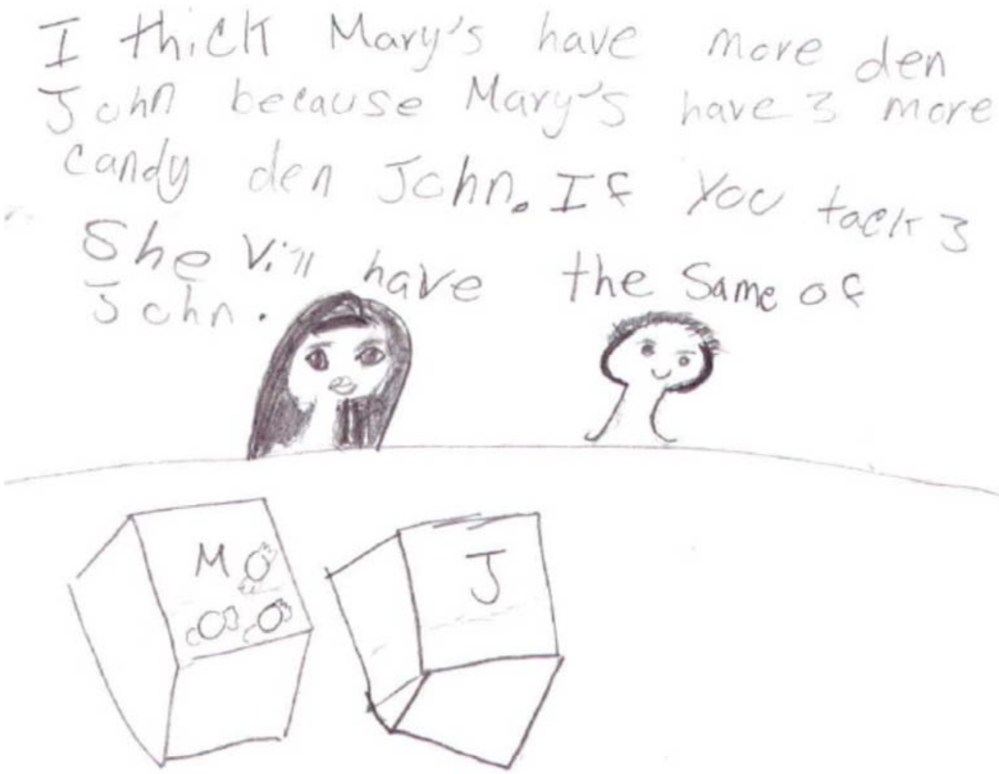


Figure 18: **Vilda leaves indeterminate the number of candies in the boxes**

Figure 19: **Felipe's question marks explicitly represent the unknown amounts in the candy boxes. The vertical lines drawn in the middle of each box are the rubber bands that hold the boxes shut.**

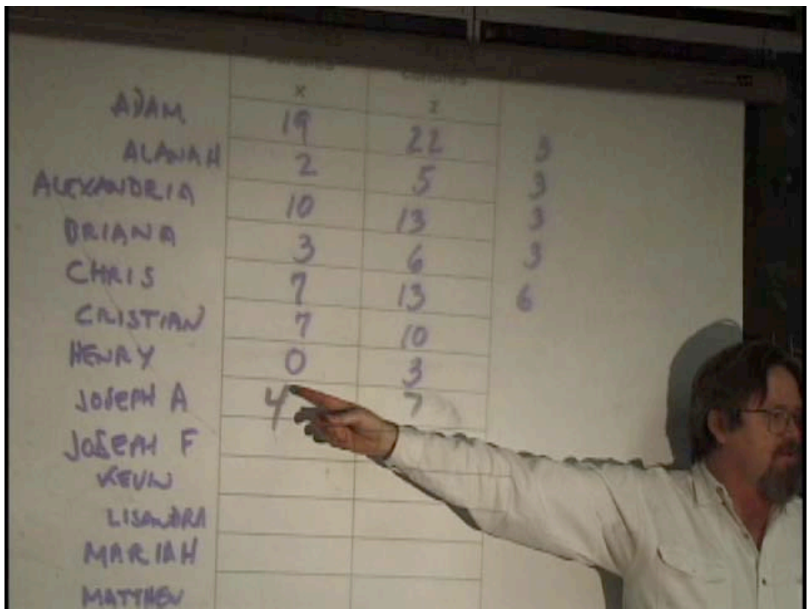
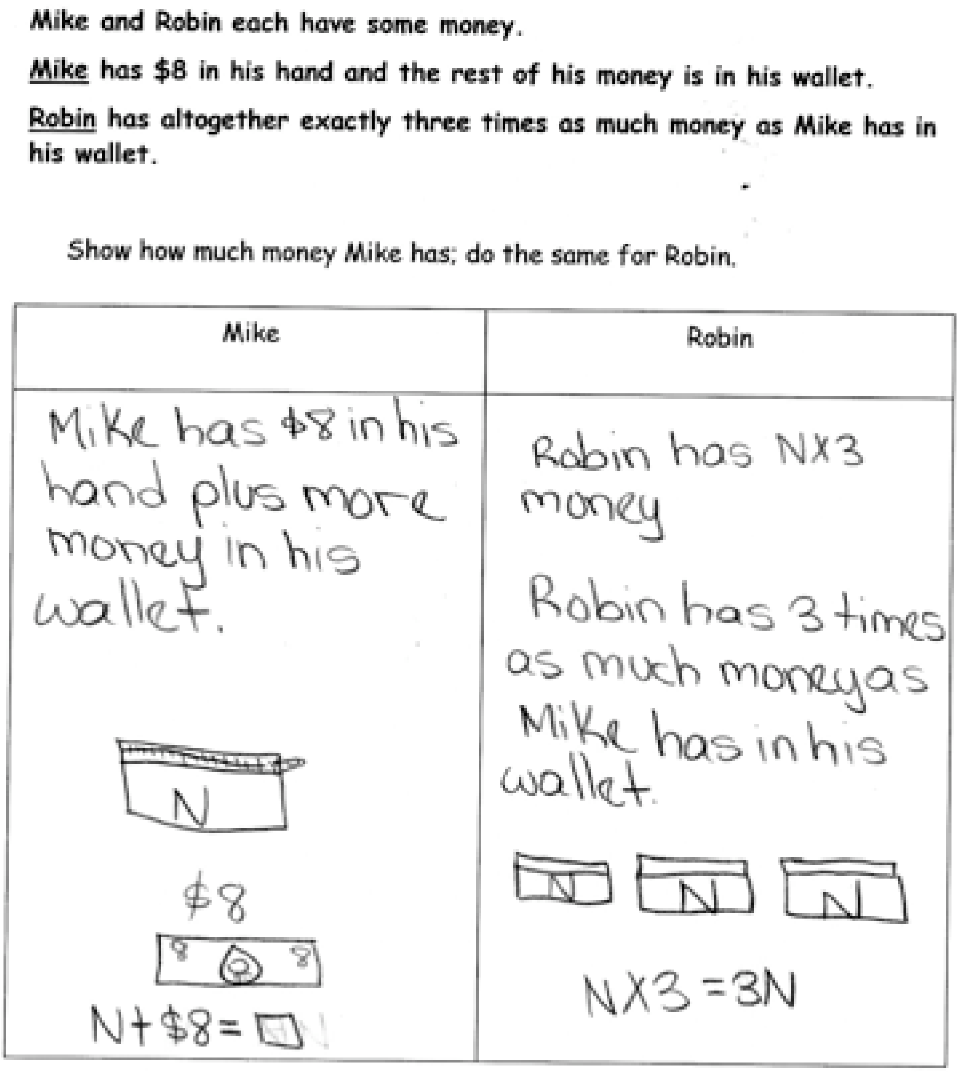


Figure 20: A table of possible outcomes. Students' names are on the left.

What can you show about John and Mary's candies? Draw or write something below.


Mike and Robin each have some money.

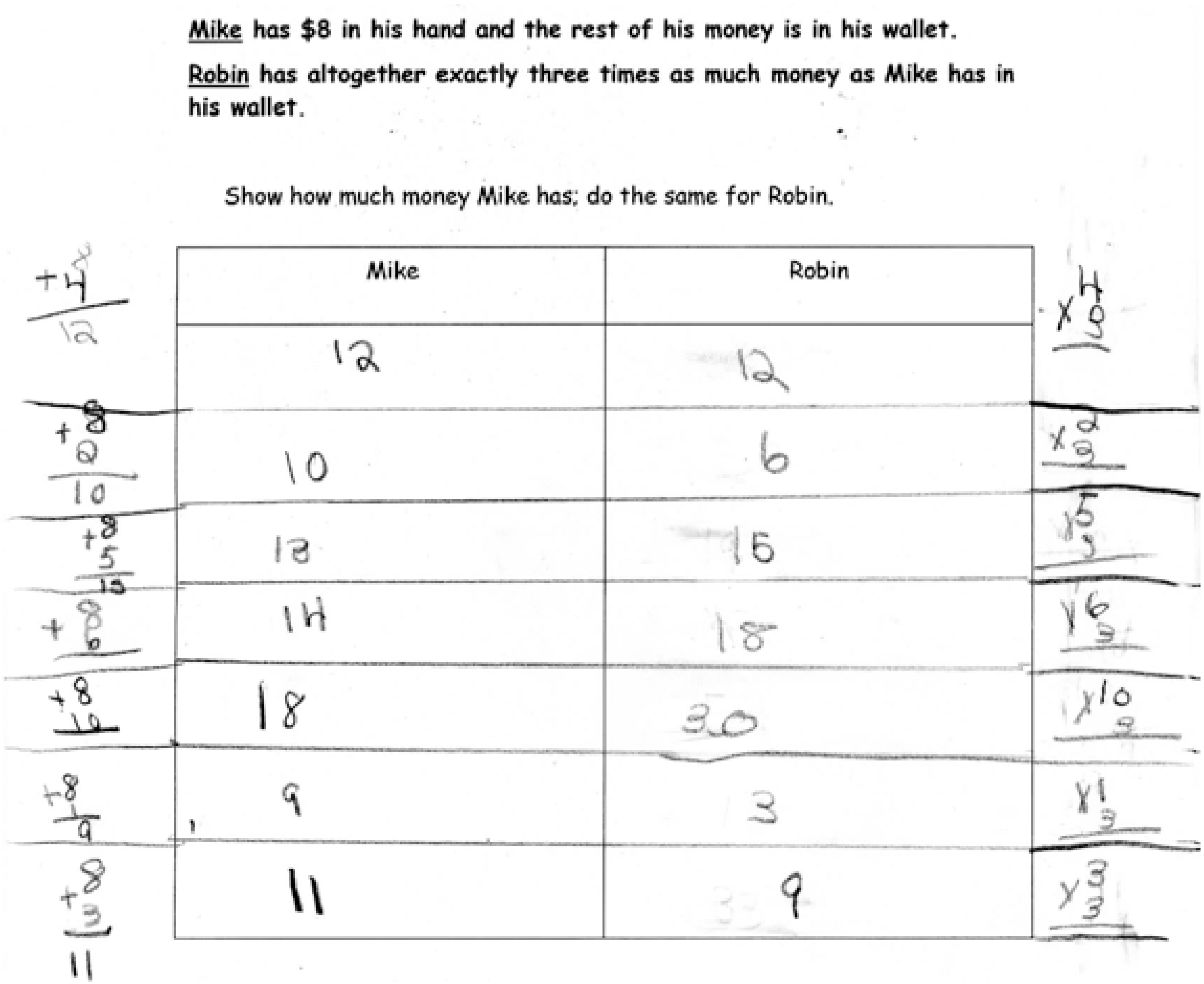
Mike has $8 in his hand and the rest of his money is in his wallet.

Robin has altogether exactly three times as much money as Mike has in his wallet.

Show how much money Mike has; do the same for Robin.

Mike	Robin
Mike has $8 in his hand plus more money in his wallet. N $8 N+$8=	Robin has NX3 money Robin has 3 times as much money as Mike has in his wallet. N N N NX3=3N

Figure 22: In the middle of grade 4, Lisandra represents the amounts of money Mike and Robin have. Note the symbolic use of the wallets with N dollars.



Mike has $8 in his hand and the rest of his money is in his wallet.

Robin has altogether exactly three times as much money as Mike has in his wallet.

Show how much money Mike has; do the same for Robin.

Mike	Robin
12	12
10	6
13	15
14	18
18	30
9	3
11	9

Figure 23: Erica decided to draw a table showing multiple possibilities for the amounts Mike and Robin had. N.B. The lines were provided by her, not given as part of the problem.

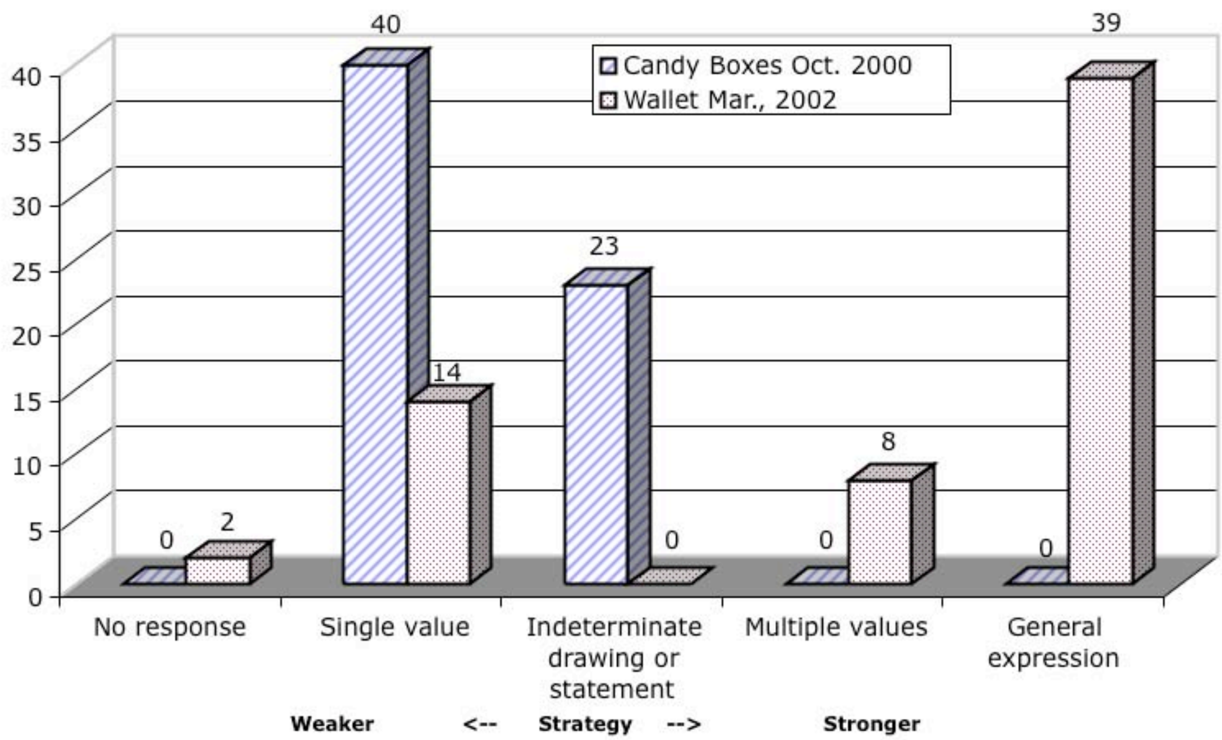


Figure 24: The Growth of Students' Thinking Over One and One-half Years.

The Wallet problem

Mike and Robin each have some money.

Mike has $8 in his hand and the rest of his money is in his wallet.

Robin has altogether exactly three times as much money as Mike has in his wallet.

Complete the table:

W	W+8	3W
In Mike's Wallet	Mike (in wallet and hand)	Robin
0	8	0
1	9	3
2	10	6
3	11	9
4	12	12
	13	
		18
		21
8		
	17	
		30
11		
	20	

Figure 25: **The table discussed by the whole class (via overhead projector). Note the use of algebraic notation for column headings.**

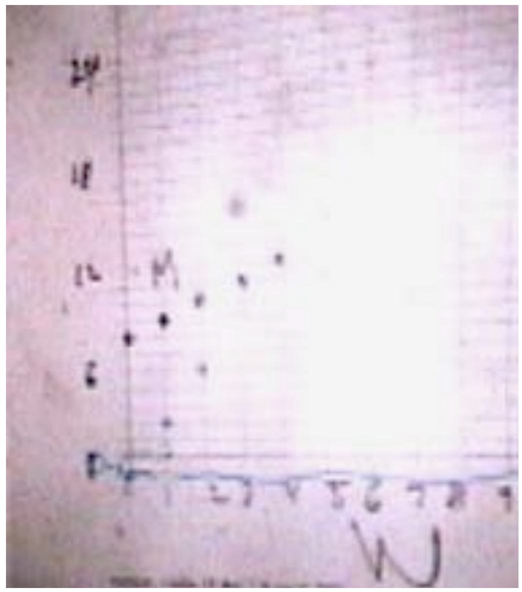


Figure 26: Mike's and Robin's amounts plotted as a function of W, the amount in Mike's wallet

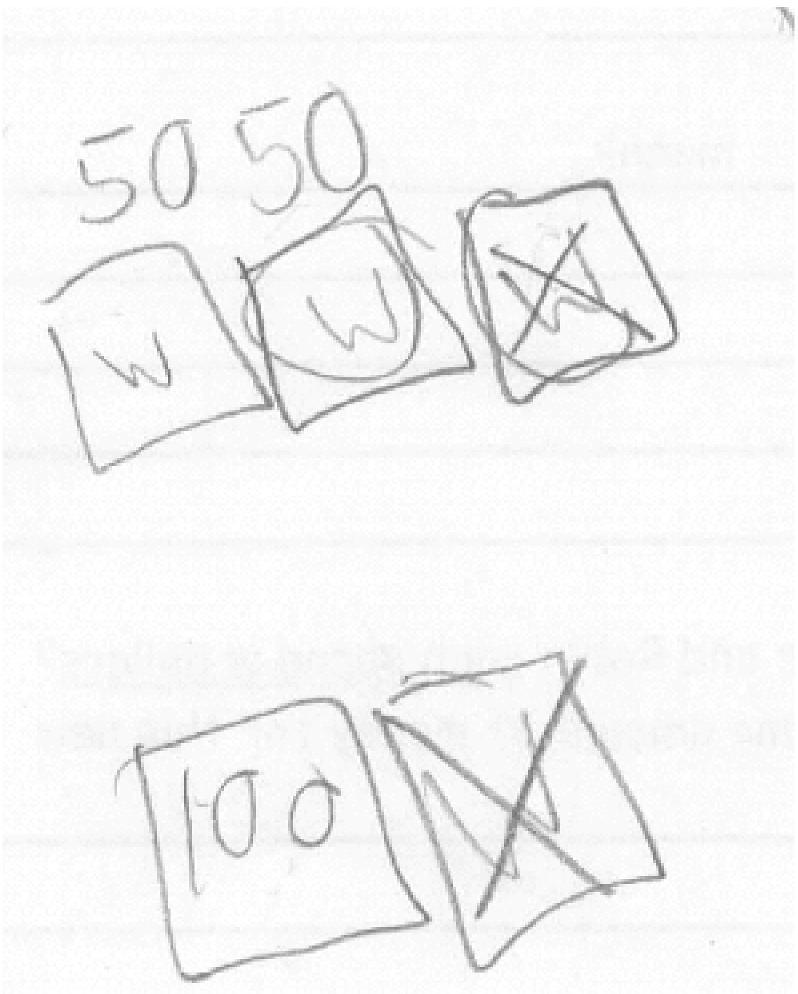


Figure 27: Jessie's representation of the 3w = 100+w and subsequently 3w-w = 100+w-w.

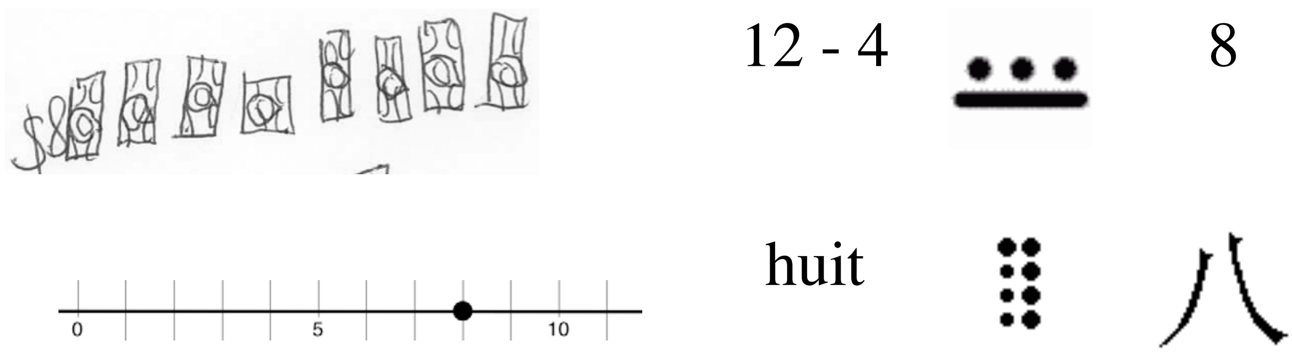
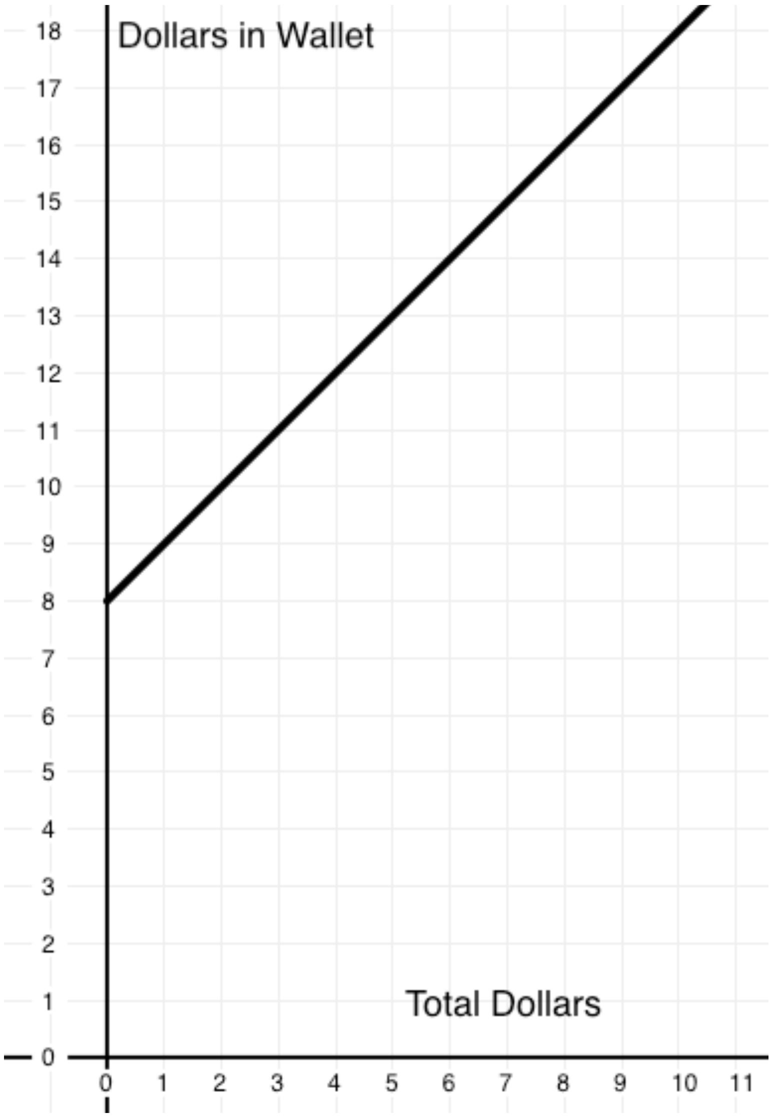


Figure 28: Various representations of the number 8: drawing, arithmetical expression, Mayan ideograph, Arabic numeral, position on number line, name in French, Braille (raised and highly raised dots that correspond, downward, to 1000_2, i.e., eight in base 2), Chinese ideograph. None is these is the number 8.

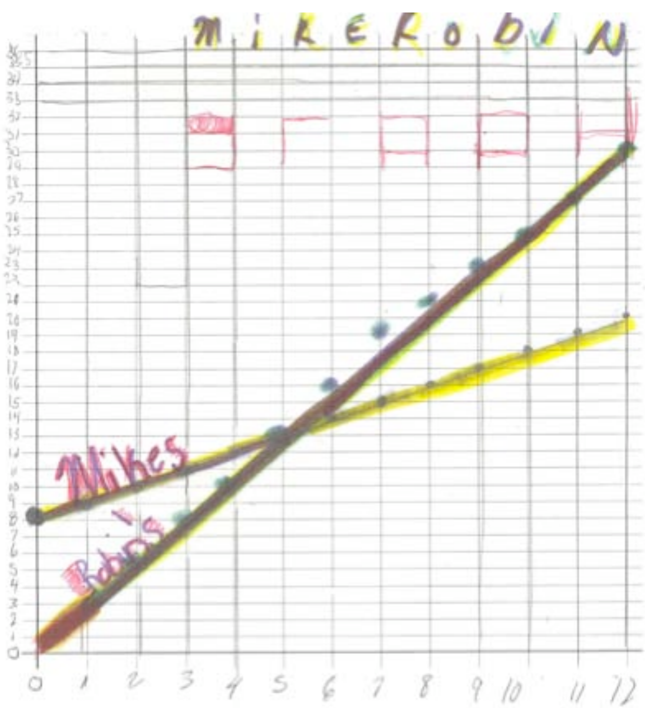


Dollars in Wallet	Total Dollars
0	8
1	9
2	10
3	11
4	12
5	13
6	14
7	15
8	16
9	17
10	18

Mike has $8 in his hand.
The rest of his money is in his wallet.

$f(x) := x + 8$, dollars

Figure 29: Representations of the same function: a graph, a function table, verbal statements, and symbolic notation.



Mike has $8 in his hand. The rest of his money is in his wallet.

Robin has three times as much money as Mike has in his wallet.

Mike and Robin have the same total amounts of money.

$$W + 8 = 3 \times W$$

Figure 30: Four representations of the same equation: a drawing (top), graphs, verbal statements, and symbolic notation.

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[2] The somewhat vague expressions, 'additive structures' and 'multiplicative structures' are widely among mathematics educators to encourage thinking about arithmetical operations as subsuming far more than the computational routines. They would emphasize, for example, that a young student may learn to multiply and divide long before showing a deep understanding of ratio, proportion, rational number, and related concepts that comprise the multiplicative conceptual field (Vergnaud, 1994, op. Cit.)

[3] Some might argue that symbols should only be introduced when students know what they mean. Were this reasoning to be applied to the case of first language learning, adults would never speak to newborns on the grounds that infants do not already know what the words mean!

[4] We use *representation* in a generic sense here to include any expression of mathematical ideas, but especially those that are observable to others and not merely private and mental. Students' own *representational forms* include natural language ("their own words"), diagrams, and written mathematical stories (although even in these cases the student relies on linguistic and graphic conventions already acquired through cultural transmission). Conventional representational forms in mathematics are those sanctioned by modern mathematicians: graphs, tables, various types of written notation, and so on. Over time students will increasingly work conventional representations into their expressive repertoires—they will "make them their own." A *representational system* (or symbol system) refers to not only the forms themselves but also to associated underlying structure and processes.

[5] Early algebra does not touch upon certain 'advanced' topics of algebra. But the qualifier, 'early,' alerts the reader to this, so there is no need to mention this as a fourth characteristic of early algebra.

[6] It is curious that Erica organized her 'data' in a table; normally a table is used to capture the results of many cases.

[7] The <u>relationship</u> between the amounts was determined, but not the amounts themselves.

[8] Clearly, N = N. But if the first N is 5 whereas the second N is 8, we produce the bothersome expression, 5=8.

[9] Their expression evokes conventional notation for a function, as in f(N):= N+3.

[10] By the time Jennifer said this, all the predicted outcomes in the table were consistent with the information given.

[11] Lisandra's Candy Boxes drawing (Figure 6) is actually difficult to classify. Is she leaving the amounts indeterminate? Or is she trying to depict specific amounts? She stated that Mary had three more than John, but she has drawn 17 in John's box and 13 in Mary's (not counting the 3 candies on top of her box). Regardless, there is little doubt that her depiction of the amounts in the wallet problem (Figure 7) is considerably more advanced.

[12] For most young children the domain will be the non-negative integers, that is the set of natural numbers including zero. Many adults will by default consider the rational numbers (those that can be expressed as a/b where a,b are integers and b is not zero), or at least the non-negative rationals, to be the default domain. Those with advanced training in mathematics tend to treat real numbers (irrationals and rationals) as the default domain.

[13] We mean here both data collected from the world as well as invented tables of numbers.

[14] This perspective leads to the unfortunate situation in which many students come to regard as equations only those equations that are analytically and symbolically soluble. Although the symbolic representation of the equation $x = \cos x$ gives no hint as to whether there are *any* solutions—and if there are, how many there are—the graphical representation of the function makes clear that there is exactly one solution and even gives a rough estimate of its magnitude.

[15] Our repeated use of the letter 'N' in instruction turns out to have some benefits. Like a researcher who follows the distribution, across the continents, of genetic markers on the y-chromosome in order to infer about the paths of human migration Wells, S. and M. Read (2002). The Journey of Man: A Genetic Odissey. Princeton, NJ, Princeton University Press. we can trace the fourth grade students' preferential use of the letter 'N' to particular discussions about the Candy Boxes task in grade three.

[16] Approximately half of the students who stayed behind repeated first grade; the rest of these dropped out of school altogether.