12

Signed Numbers and Algebraic Thinking

Irit Peled
University of Haifa

David W. Carraher
TERC

We suggest that signed numbers[1] and their operations belong in early grades. If carefully introduced, signed numbers can make fundamental algebraic concepts such as equality and function accessible to young students. In turn, negative numbers and operations with signed numbers can be learned more meaningfully when taught within an *algebrafied* curriculum.

We first identify some of the problems related to the learning of signed numbers. Then we show how algebraic contexts can facilitate the learning of this problematic topic. Finally, we look at how signed numbers provide a supportive context for learning algebra.

[1]Signed numbers are positive and negative numbers. They refer to integers, not merely the natural numbers or positive integers. They refer to rational numbers, not merely the non-negative rational numbers. They refer to real numbers, not merely non-negative real numbers. Students may learn to accept negative numbers in the co-domain (output of computations) before accepting them in the domain (input of computations). When they are comfortable with both, we say they have learned not only that signed numbers exist, but they can serve as the input for addition, subtraction, multiplication, and division as well as other functions.

The present discussion of signed numbers and algebra builds on mounting evidence suggesting that young children can learn algebra (Blanton & Kaput, 2000; Brizuela & Earnest, chap. 11, this volume; Carpenter & Franke, 2001; Carraher, Schliemann, & Brizuela, 2001; Carraher, Schliemann, & Schwartz, chap. 10, this volume; Davydov, 1991; Dougherty, chap. 15, this volume; Schliemann et al., in press).

USING DIDACTICAL MODELS TO CONSTRUCT A MATHEMATICAL MODEL

In working with children to construct meaningful algebraic structures, keep in mind that we are not merely creating mathematical models to "play around with functions." One of our main goals is to make the emerging algebraic structures available to and actively used by children in analyzing and modeling situations. The algebraic concepts will serve as mathematical models, namely, tools with which different phenomena can be conceived and organized (Gravemeijer, 1997; Greer, 1997; Shternberg & Yerushalmy, 2003).

Teachers use *didactical models*—that is, manipulative materials used in a specifically defined language within a planned teaching trajectory. The teaching trajectory might employ a sterile model such as Cuisenaire Rods, a situation model such as the Realistic Mathematics models (Gravemeijer, 1997), or anything in between as a means of helping children construct their toolbox of *mathematical models*. These mathematical models are then applied in solving problems, and their conception is changed and expanded by the application process. Figure 12.1 represents the construction and application of the mathematical model. The double-headed arrow drawn between the application and the mathematical model stands to convey that even when one applies an already acquired (a somewhat misleading term) mathematical concept, the concept's image keeps changing and expanding following each application.

We discuss the learning of signed numbers within the general framework provided by Figure 12.1, focusing on situations involved with the construction and the application of the mathematical concepts of signed number operations. We give several examples of situations that, according to our analysis, can affect the senses and constructs of these concepts and their predisposition to become activated modeling tools.

There are different kinds of didactical models for teaching signed number operations (Janvier, 1985). One favorite among teachers in Israel is the Witch model, which involves adding or taking away warm cubes or cold cubes to and from the witch's potion bowl. Similar models can be found in U.S. textbooks. For example, Ball (1993) mentions a "Magic-Peanuts model," which

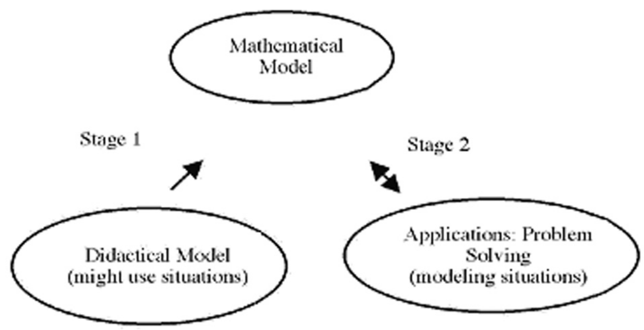

FIGURE 12.1. Mathematical models, didactical models, and situations.

she suspects might create the impression that mathematics involves some kind of hocus-pocus. Indeed, it is doubtful whether such a model would facilitate future use of signed numbers in modeling realistic situations.

Some teachers appear to believe that children will find it easier to remember a set of strange rules in such context better than a set of number rules. In a course for preservice secondary school teachers, students interviewed teachers about models they use for teaching operations with signed numbers. One of the interviewed teachers explained: “I use a model with piles and ditches. The minus sign stands for a ditch, the plus sign stands for a pile. Subtraction means: cancel. Addition means: put more. So, if I have –(–3) it means that I get a cancellation of a ditch that is 3 m deep, it means that I have a 3 m pile. . . .”

If performance is evaluated strictly in terms of computational fluency, then perhaps all these models have a similar effect and just teaching the rules would not be much different. Indeed, Arcavi (1980) showed that there is no difference in computational performance between four different instructional models for signed number multiplication.

However, signed numbers entail more than computational skills. They are supposed to become an addition to the set of mathematical lenses with which we model problem situations. Thus, students need to construct these mathematical models; teachers need to introduce these abstract models through didactical models. By and large, didactical models for signed numbers have not been very successful. Pitfalls associated with money models are discussed in one of the following sections. A promising direction emerges from Realistic Mathematics Education (RME; Gravemeijer, 1997). Based on the RME approach, Linchevski and Williams (1999) have constructed and tested several models, analyzing where the models work and where they fail. We would like to suggest

Table 12.1
Word Problems that Pre-service Teachers Created for the Expression "2-7."

Problem / *Context*	*Consistent with "2-7"*	*Consistent with "7-2"*	*Incomplete answer (no question asked)*	*No answer*	*Total*
Money/debt	2	6	1	–	9
Temperature	1	–	–	–	1
Height	1	1	–	–	2
No context (no answer)	–	–	–	3	3
Total	4	7	1	3	15

that a combination of an RME approach with algebraic tasks (in the spirit of the examples in the following section) may offer an even more promising direction.

After being introduced to signed numbers and operations with signed numbers, children are expected to apply their new tools in solving a variety of arithmetic problems. Unfortunately, as we show, most of these problems do not facilitate conceptual growth. The following section illustrates that algebraic problems are more suited than arithmetic problems to promote meaningful learning of signed numbers.

The Challenges Posed by Signed Number Problems

Before we criticize the common signed number textbook problems, we should admit that it is not easy to compose good-signed number problems. With some exceptions (Rowell & Norwood, 1999), children and teachers do not have much trouble finding everyday situations that correspond to expressions such as 3 + 4 = __ or 4 × 5 = __. However, they are often at a loss for finding contexts involving negative numbers and measures—for example, when trying to write a story for 2 – 7 = __ or 7 – (–5) = __. Temperature and money (credits and debts) are favorite examples, and yet even these contexts pose special challenges. For example, in composing a word problem to exemplify the expression, 3 – 5 = –2, a student may suggest, "Johnny had three apples and he had to give five apples to his friend, so now he has minus two apples." The student is able to employ the negative numbers, but only in a contrived, artificial way. Table 12.1 summarizes the answers 15 preservice teachers gave when asked to compose a story problem for the expression, "2 – 7 = __."

The table shows that:

- Most preservice teachers (9 of 15; 9 of the 12 who gave answers) used a money/debt context. An example of a reasonably appropriate problem (one that can indeed be solved by 2 – 7) that a teacher wrote is the following: I have $2. I owe you $7. How much money do I [really] have? (A somewhat clearer, although awkward, version of the final question would be: How much do I really have, considering my assets and debts?)
- Only three preservice teachers used an alternative context. Of those, two teachers composed appropriate problems using contexts dealing with measurement of heights and temperature. For example, one teacher wrote: In the middle of the winter my thermometer read 2°. Overnight it dropped 7°. What temperature was it the next morning?
- Debts were used in a pseudonegative role. Consider the following example: If Jennifer owes Matthew $7 but she only has $2 to give him, how much money does she still owe Matthew? While being composed as a problem that can be solved using 2 – 7, this problem is more likely solved by using 7 – 2. This was confirmed when preservice teachers, given some of their own word problems, solved them by using 7 – 2 rather than 2 – 7.
- Some of the preservice teachers' word problems made use of the measures given, but included additional assumptions that went beyond the data given. Consider the following word problem created by a preservice teacher to reflect the expression, 2 – 7.
  Steve has $2, but owes his friend $7, if Steve pays back his friend, how much money will he have?
  We wonder: How can Steve pay back his friend $7 if he only has $2?

How People Avoid Signed Numbers

Part of the difficulty in creating signed number tasks stems from the fact that many problems that come to mind can be solved without negative numbers, relying instead on work-arounds.

Mukhopadhyay, Resnick, and Schauble (1990) compared children's performance on problems posed in the context of a story with their performance in number problems (calculations) that according to the authors' conception, correspond to the contextual problems. The authors found that children's performance "is far more complete and competent" in a narrative story about a person whose monetary standing goes up and down over time than in what they term "isomorphic problems presented as formal equations with mathematical notations." It may thus appear that the everyday context helps in dealing with signed numbers.

However, for the children these were not isomorphic cases. Although the story situations could be matched by experts to signed number expressions, in dealing with them children did not use negative numbers to add up debts; rather, they performed simple addition and subtraction of non-negative quantities.

In interviews with strong and weak-performing sixth graders a year after they learned addition and subtraction of negative numbers, the first author found that students exhibited overall low computational performance (Peled, 1991). In another unreported part of the study, the children were given word problems that could be solved using signed number computations. Most children solved the problems correctly while circumventing signed number operations.[2]

One of the word problems in the study involved a context often used in textbooks to teach the concept of the difference between signed numbers: elevation with respect to sea level. The children were asked to find the difference in height between two cities—one located below sea level, at –200 meters and the other above sea level, at +300 meters. Rather than subtracting the numbers to find the distance by: 300 – (–200) = 500, the children simply added the (absolute values of the) distances from sea level: 300 + 200 = 500.

Students who correctly solve sea level problems show: (a) They understand the directed nature of the measures in the story context but (b) they have not mastered, or are not yet fully comfortable with, signed numbers. To appreciate the significance of the first point, it is important to recognize that the young child views numbers as counts: The natural numbers are used by them exclusively for representing the cardinality of sets (how many?). Extending the concept of number to include measures (how much?) is a major achievement. Students who solve the sea level problems not only treat numbers as measures. They also display a careful distinction between above and below measures, similar to the distinction between assets and debts, in the case of money problems. Expressed another way, they exhibit some understanding of two measure worlds separated by the zero point, similar to the divided number line model suggested by Peled, Mukhopadhyay, and Resnick (1989). But, eventually they will need to extend this conception further allowing for a number to possess two characteristics: measure and direction. They also need to regard the number system as a single coherent system (rather than two separate worlds) with unified operations that hold regardless of the sign of the numbers.

---

[2]There is nothing inherently wrong with solutions that do not use signed numbers. However, if our goal is to increase understanding of expressions such as 3 – (–5) and the conditions for their application, we need to look someplace else.

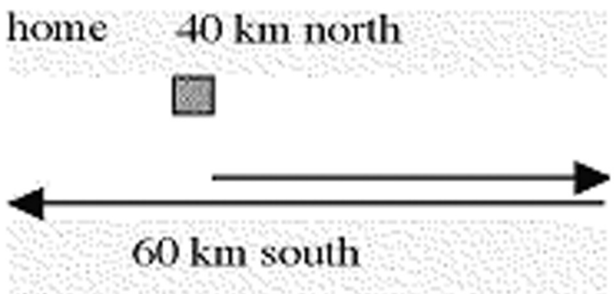

FIGURE 12.2. Trip A.

LEARNING ABOUT NEGATIVE NUMBERS IN AN ALGEBRAIC CONTEXT

Whereas arithmetic problems present a relatively simple *specific case*, algebraic problems can introduce a challenge that calls for using mathematical tools to model *general algebraic structures*. Two examples involving trips serve to illustrate.

Algebraic Tasks for Learning Signed Numbers

Trips Along a Straight Line

Trip A: An Arithmetical Trip. Anne drove 40 kilometers north from her home to an out of town meeting. She then drove back going 60 kilometers south to another meeting. After both meetings were over, she called home asking her husband, Ben, to join her.

a. How far will Ben have to go and in what direction?
b. Write an expression for finding the length of Ben's trip.

A graphical diagram of the trip would look like the drawing in Figure 12.2.

From looking at Trip A (Fig. 12.2), one can see that Anne went south more than she went north, and Ben will have to travel that excess amount, namely, 20 kilometers south. The symbolic expression that best reflects the actual operation is:

Equation 1: An expression for Trip A

$$60 - 40 = 20$$

The symbolic expression teachers might have had in mind in composing such a problem is shown in Equation 2:

Equation 2: What teachers may have thought of for Trip A

$$40 - 60 = -20$$

As discussed earlier, one can avoid Equation 2, opting instead for Equation 1 while mentally keeping track of the direction. The following version, however, makes it harder to get away with this "partially explicit, partially implicit" approach.

Trip B: An Algebraic Trip. Anne drove a certain number of kilometers north from her home to an out of town meeting. She then headed 60 kilometers due south to another meeting. After both meetings were over, she called home and asked her husband, Ben, to join her. How far will Ben have to go and in what direction?

a. Write an expression for the length of Ben's trip.
b. Could Anne have driven less than 60 kilometers north on her first trip? If not, explain why. If she could have, give an example and explain its meaning.

Trip B is more general than Trip A: The initial part of Anne's journey corresponds to a mathematical variable. Using a number line representation with Anne's home marked as 0 and the direction to the right as distance in km due north, a typical figure might look like the drawing in Figure 12.3.

One legitimate expression for the husband's trip would be "$X - 60$ kilometers north." Notice that if x, the distance Anne first traveled northward is greater than or equal to 60, the answer is non-negative ("Go north! or if $x = 60$, stay put!"). If x is less than 60, the answer will be negative ("Go south!"). In principle, this should not be difficult. However, many solvers neglect to check the constraints on this expression. Question B was designed to serve as a hint, explicitly raising the possibility that $X - 60$.

The minus sign retains the same"ol" definition (moving left on the number line, as depicted in Fig. 12.4), even when the end point is a negative number. If, for example, $X = 40$, then Ben will have to travel 20 km south, as shown by Equation 3.

Equation 3:

$$X - 60 = 40 - 60 = -20$$

This simple description, $X - 60$, holds for a wide variety of cases. Furthermore, one can check its boundary conditions, investigate different cases using informal knowledge, and then discover generalizations and connections to formal knowledge. In this sense, the algebraic nature of the expression facilitates an understanding of operations with signed numbers.

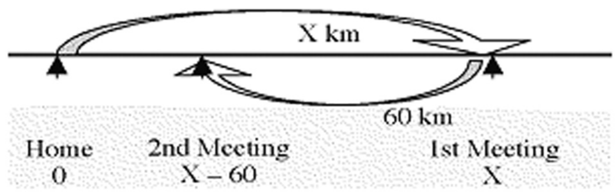

FIGURE 12.3. Version 2 of the "On the Road" problem.

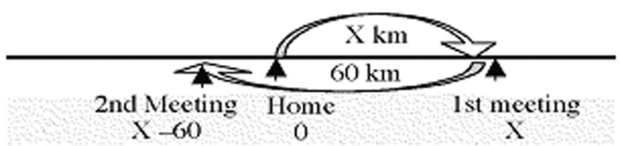

FIGURE 12.4. The "On the Road" problem for x < 60.

Changes in Temperature

Temperature Story 1: Arithmetical. On the February 24th last year the morning temperature was –5° and on the same date this year the morning temperature was 3°. What was the change in temperature from last year to this year on February 24th?

The problem can be solved with the circumvention strategy used in the elevation above sea-level problem described earlier. In the sea-level problem, children added the absolute differences in height from the two cities to zero (one city was below sea level and the other above sea level). Here, one can add the absolute differences around zero temperature: 5 and 3, or consider going up to zero and then up from zero to 3°. The total change amounts to 8°. Accordingly, one would conclude: On February 24th this year, the temperature was by 8° higher compared to the same date last year.

The symbolic expression that best reflects the solution is: $5 + 3 = 8$. However, the expression the teacher was probably hoping students would use was: $(+3) - (-5) = 3 + 5 = 8$. Let us now see why a more general framing of the problem is likely to encourage students to use signed numbers in their expression.

Temperature Story 2: Algebraic. A computer with a special measuring device records the morning temperature in your office each day at a specific hour. As a meteorologist, you want it to calculate the difference in temperature between the same day last year and this year's record. The computer holds the data, but you have to tell it how to make the calculation. What would be your instructions? Here are the temperature data of the last 2 weeks and the corresponding data from last year.

The strategy one uses in solving this problem depends on one's experience with algebraic problems. An expert solver can use a top-down solution. He would choose some variables, such as T1 to denote the temperature on a given date last year, and T2 to denote the temperature on the same date this year. Then express the change in temperature from last year to this year (on a specific date) by the generalized expression: T2 – T1.

Depending on his experience, a problem solver might check if the expression is valid for negative as well as positive numbers. For example, if the temperature on a chosen date last year was –5 and the temperature on the corresponding day this year is 3 (as in Version 1), the change in temperature can be calculated informally as in version 1 and then compared to what one gets by substituting the variables in the general expression: T2 – T1 = (+3) – (–5) = +8, meaning there was an increase of 8°.

A novice has to search for patterns, work bottom up, organize the situation, and make generalizations. Specifically, a novice would look at a sequence of temperature changes, and come to recognize the advantages of using signed numbers to differentiate between temperature decreases and temperature increases. The task requires the use of variables, generalization of an expression, awareness of different possible cases, and use of available mathematical tools, in this case, the use of operations with signed numbers. These are exactly the skills we want a novice to develop.

Our analysis is theoretical and should, of course, be tested by refining and implementing the tasks. Yet, there is some evidence that an effective learning trajectory can be designed. Researchers from the Early Algebra Early Arithmetic Project (Carraher et al., 2001; Carraher et al., chap. 10, this volume) have shown that third-grade children in the Early Algebra Project can express relations between heights and compare differences in cases where heights were variables and not specific values.

A Word of Caution About Money Contexts

As noted earlier, in-service and preservice teachers often use money and debts to construct signed number problems. The assumptions are that this context is meaningful to children and that children will use signed numbers in modeling the situations.

As mentioned earlier in discussing the study by Mukhopadhyay et al. (1990), children solve money problems by using their knowledge about the situation circumventing the need to use signed numbers. Still, this study showed that children have some understanding of debt situations and might lead us to conclude that it would be helpful to find problems that use this context and require the use of signed numbers (e.g., by using an algebraically structured context as discussed earlier).

However, even a very familiar context can be tricky. The following, previously unpublished, episode (TERC Tufts Early Algebra Early Arithmetic Project, 2000) exemplifies the difficulty of mapping situations involving money to a mathematical representation.

Two children, Filipe and Max, are enacting mathematical expressions as movements along a number line at the front of the classroom. The number line is represented as a clothesline with the integers, –10 to +20, written on labels hanging by clothes pins and spaced approximately one foot apart. Filipe and Max positions are to indicate how much money they have; their displacements are to signify the spending or obtaining of money.

Here is a brief synopsis of the episode. At a certain point, Filipe and Max are told they have each spent $3, which they show by moving three units leftward: Max moves from 8 (i.e., $8) to 5 ($5), and Filipe from 3 ($3) to 0 ($0). The teacher then informs them they are each to buy another item (a hamburger) costing $2. Max moves correctly to 3, as depicted in Figure 12.5. Filipe appears to be puzzled by the fact that he has no money left to pay for the item. The teacher offers to lend him $2.

Before reading the transcript of the episode, the reader is asked to consider the following question: Where should Filipe stand on the number line after receiving the $2 loan from his teacher to purchase the hamburger?:

- At +2? (He holds the $2 loan in his hand.)
- At 0? (The money is not his own.)
- At –2? (He owes the teacher $2.)

All three possibilities arise in the following excerpt:

Barbara [the teacher]: . . . and they spend another $2. Where would they end up? . . . Max, where would you end up?
Max: [moving to three]
Barbara: And where would . . . where would Filipe end up?
Filipe: Washing dishes if I didn’t pay. [Class erupts in laughter]
Barbara [Repeating his words]: Washing dishes if you didn’t pay.
Barbara: But where . . . where would you be on the number line?
Filipe: I’m at it. [Filipe is still positioned at zero.]
Barbara: You’re at it. So you would *stay at zero*?
Filipe: Yea.

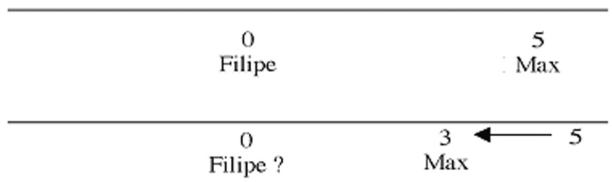

FIGURE 12.5. Number line displacements by Filipe and Max.

Barbara: How come?
Filipe: Because I have no more money.
Barbara: You have no more money. Well, Anne [Barbara notices that Anne, a member of the research team, has something to say.] . . . yea?
Anne: What if he borrowed $2 from somebody?
[Max pretends to offer $2 to Felipe.]
Filipe [accepting the offer]: Thanks Max.
Barbara [concerned that this offer by Max will require that he reposition himself]: Oh, but what if, if Max lends you $2, where will Max end up at?
Student: The zero. [Another student: One.]
Barbara: At one. So *I* am going to lend you $2. Okay? Max, you stay at three.
Barbara: <u>I</u> will lend you $2.
Filipe: [accepts the imaginary $2 and moves to +2]
Barbara [addressing the whole class]: Do you think he should go up to two? Does he actually have $2?
Students: No . . . Yes.
Barbara [to Filipe]: How much money do you have of your own . . . your own money?
Filipe: [moves back to zero, apparently interpreting Barbara's question suggesting that he should not have moved.]
Barbara: Okay. And he *owes* me $2.
Barbara: Okay. Are you all going to keep track of that money?
Students: Yes.
Barbara [realizing that the information about the loan is not apparent from Filipe's position]: You have to all be my witnesses. He owes me $2. Where would he go?
Filipe [looking for a practical solution]: I'd get a job.
Barbara: You'd get a job, but where would . . . if I had to show with numbers . . .
Ariana: *Stay at the zero*!
Barbara: Ariana, that's actually very good because that's how much money . . . that's how much money he has.
Barbara: He doesn't have any money, but he owes me $2 and we should have to show that, in some way, on the number line.
Filipe: [moves to –2]

At first Felipe feels no need to move to the left of the origin because the least he can have is zero dollars [6–11]. Barbara, the teacher, had expected him to move to –2 under the assumption that the number line was being used to register his balance of credits and debits. Anne suggests [13] that someone lend Felipe $2, presumably to keep him solvent and make it easier for Felipe to conform to Barbara's expectations. When Max graciously offers $2 to Felipe [14–15], Barbara realizes that this would affect Max's amount [16] and require that he move; so she makes the loan [18–19] herself.

The matter is still not settled. How should Felipe respond to the fact that he has received the loan? Should he move to another position on the number line? Or stay at zero? Felipe and several classmates believe [20–22] he should move from zero to +2; after all, he now holds $2 in his hand. Ariana thinks [31] that Felipe needs to stay at zero. Barbara acknowledges the reasoning underlying Ariana's answer [32] "because that's . . . the amount of money [Felipe] has" in terms of his balance.[3] (Felipe holds a different, more optimistic, view of "how much money he has."). Yet, Barbara's successive comments about Felipe's debt [33] together with her previous effort to enlist the class as her witnesses [28] show that she still believes he should move to –2, and eventually he gives in and takes the hint [34] moving to –2.

As we can see, the situation is quite complex and a teacher who thinks in terms of debits and credits may fail to recognize when a student takes into account only the tangible assets that are present on his person. As a matter of fact, not only young students choose to focus on the latter. The following example shows that the situation can be confusing and ill-defined for adults as well.

We asked 15 preservice teachers to think about the following situation:

> Let's imagine that you are an obsessively organized person and you keep track of the exact amount of money that you have in your checking account and wallet altogether. Every night you record the amount of money you have by writing the relevant number in your calendar.
> One day you have no money left and you visit your parents and borrow $20 from them. What number will be recorded that night in your calendar? Explain.

The variety of answers in Table 12.2 speaks for itself. Even adults, including teachers, sometimes get confused in this context. We conclude by saying that the money context is not straightforward and does not automatically make signed numbers more accessible.

---

[3]This is a reasonable conclusion if Felipe has not yet purchased the hamburger.

Table 12.2

Pre-service Teachers' Answers for the Money Recording Problem.

Answer	*Explanation*	*N*
–20	Because even though I have no money I cannot write "zero" since I owe them twenty dollars. Therefore by writing –20 I am accounting for what I have to pay back. It doesn't belong to me and I will need to give it back.	8
20	I will record $20 on the calendar because I have $20 in cash, even though I owe this to my parents who never ignore a debt. Twenty will be recorded that night on the calendar. You record the total of what's in your account and your wallet, so since you have $20 in your wallet that is what you record.	6
20 or –20	–20, because it was borrowed, I still owe that money, I'm in debt. Maybe I'm 20 because that's what I <u>have</u>.	1
0	During class discussion one of the students suggested 0 as another possible option.	

LEARNING ALGEBRA IN THE CONTEXT OF SIGNED NUMBERS

We have tried to show that an algebrafied curriculum offers good opportunities for meaningfully introducing signed numbers. This section argues that signed numbers can benefit the learning of algebra. Our claim rests on the idea that fundamental algebraic concepts can be enriched and more abstract concept images can be developed.

Functions and Graphs

Functions are an important topic in mathematics—arguably, the very cornerstone of an algebraic curriculum. Basic functions such as $f(x) := x$, $f(x) := x + 3$, and $f(x) := 4x$ play important roles in elementary mathematics even though they are likely to be implicit in discussions about matching expressions, adding, multiplying, and the like. The examples in this section demonstrate the importance for algebra of extending the number system to include signed numbers. The principal idea is that certain

Table 12.3
Three Inequalities (Comparisons of Functions).

Expression 1	2x ? x
Expression 2	X + 3 ? X
Expression 3	X + 3 ? 3

Table 12.4
Truth Table for the Three Inequalities in Table 12.3

When x	*Expression1*	*Expression 2*	*Expression 3*
0	True	True	True
4.5	True	True	True

FIGURE 12.6 Comparing graphs (take 1).

questions concerning algebra can be more suitably explored when negative numbers are taken into account.

Consider the following problem: What can you say about the following expressions? Are they valid? Under what conditions? (See Table 12.3.)

Table 12.4 and Figure 12.6 show some of the conditions under which the three inequalities hold. It might seem that the inequalities are generally true. Unfortunately, this picture is misleading. The graphs in Figure 12.7 demonstrate that the constant function $f(x)$: = 3 is greater than $x + 3$ for $x < 0$.

Likewise, the function $2x$ is greater than x only for $x > 0$. The issue goes beyond extending the domain and co-domain of functions. In the case of $f(x)$: = $2x$, it means that "multiplying by two does not always result in a greater number." So it provides evidence that can potentially challenge the belief (Greer, 1992) that "multiplying always makes bigger and division makes smaller." The extended number system thus supports a more complete knowledge of basic functions such as x, $x + 3$, and $2x$.

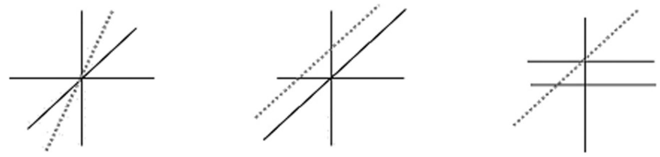

FIGURE 12.7. Comparing graphs (take 2).

Comparing Functions

The following problem was given to preservice secondary school mathematics teachers, to graduate students in mathematics education, and to a few mathematics education experts.

Comparing Elevations With Respect to Sea Level.

Three friends are vacationing in a resort that has beautiful mountains and impressively deep canyons. One day they were conducting a conference call in order to decide about a meeting place for lunch. Before making a decision, they informed each other where they were by indicating the height of their location in sea level terms (don't ask us how they know it . . .):

Anne tells her friends her current elevation.
John says that his elevation is 50 meters more than Anne's.
Sophia says that the height where she stands is 2 times that of Anne's.

a. Who is standing at the highest place? Explain.
b. Is it possible that John and Sophia are at the same height?
c. Is it possible that Anne stands at a higher place than Sophia?
d. Did you use a graph to answer the above question? Yes/No

If you did not, try to use a graph now, explain if it supports or changes your previous answers.

This problem can be solved by drawing the functions describing the height relations. If we represent Anne's height by X meters, then John's height is $J(x) = X + 50$, and Sophia's height is $S(x) = 2X$. In order to make height comparisons we need to represent Anne's height by using the identity function $A(x) = X$. Because elevation at sea level can also be negative, the functions should be drawn for any x ? R, getting the drawing presented in Figure 12.8.

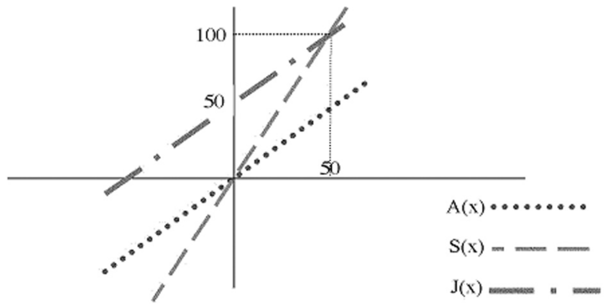

FIGURE 12.8. The sea level elevation problems.

Some of the characteristics found in the solutions suggested by the different problem solvers (preservice teachers, graduate students, and mathematics education experts) were:

- Most of the problem solvers did not use a graph in the process of investigating the different possible cases in this situation. This population included secondary school teachers who were teaching linear graphs at that time.
- When asked to draw a graph many of the solvers had trouble realizing that in order to compare $2X$ with X one should draw the identity graph $f(x) = X$ and not use the X axis as a representation of X. . . .
- When asked to draw a graph many of the solvers ignored the interval $x < 0$ and made limited comparisons.

These findings support the present argument about the need to work with situations that offer opportunity to investigate many cases and enhance fluency in making comparisons. Another conclusion relates to the use of graphs. Many of the problem solvers showed that graphs had not become a natural tool for them in their investigation. Apparently, they use graphs only when asked to do so, only for certain familiar types of problems, and with specific instructions on the choice of axis.

Going back to our original argument, the extension of the number system opens up possibilities for constructing problems with rich investigations that can promote the analysis of functions and through the use of graphs.

Composing Transformations

When numbers have a sign, the sign can assume the sense of direction, and the numbers can be represented as vectors rather than points on the

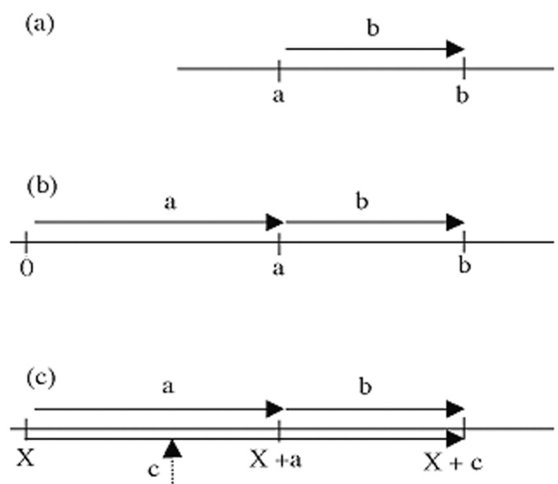

FIGURE 12.9. Defining a + b as one or two displacements.

number line. In the expression $a + b = c$, "a" can be perceived as a starting point, transformed by the function "+ b" (a unary operation with the operator + b) to point "c," as depicted in Figure 12.9a.

The expression can also be perceived as a sequence of displacements transforming 0 to the point "a" (thus $+a$ is both an operator and a point) and then transforming "a" to the point "c," as shown in Figure 12.9b.

Several authors have argued that children need to think of the operations as operations on transformations (Janvier, 1985; Thompson & Dreyfus, 1988; Vergnaud, 1982). Symbolically: $X + a + b = X + (a + b) = X + c$ or $(+a) + (+b) = +c$, as represented in Figure 12.9c.

Vergnaud (1982) suggests that some cases can be perceived as a composition of two transformations that yields a third transformation. For example, he describes a situation where: Peter won 6 marbles in the morning. He lost 9 marbles in the afternoon. Altogether he lost 3 marbles. This situation is modeled by: $(+6) + (-9) = (-3)$ (Vergnaud draws the + sign differently to denote that it stands for addition of signed numbers). In situations like these, the problem solver will develop a sense of signed numbers as changes, similar to the meaning suggested by others (Davis & Maher, 1997). Although there is no information on the initial amount or the final amount, one can figure out the total change, which would be valid for any starting point.

In the following sections, the implications of these meanings are discussed in terms of their possible contribution to algebraic concepts.

Broadening the Meaning of Equals

Extensive practice with addition and subtraction leads children to develop certain primitive conceptions of equality. Children in primary grades tend to view the two sides of the equal sign nonsymmetrically: The left side is taken as a request to carry out an operation, the result of which is displayed on the right side. Accordingly, they view expressions such as $8 = 3 + 5$ as illegitimate (Carpenter & Levi, 2000; Filloy & Rojano, 1989; Kieran, 1981).

Equality expressions in early grades involve mainly addition and subtraction with amounts represented by natural numbers and quantity relations that obey the part–part–whole structure. As suggested by Freudenthal (1983), this structure can quite naturally (although not easily) be extended to include fractions.

The extension to negative numbers is a different story. Both the meaning of a number and the meaning of operations defined on the numbers have to undergo a drastic change. The image of a number as representing a physical quantity or a measurement has to change or at least get a new dimension that differentiates the quantity –2 from +2. The new constructs have to account for the order relation according to which –4 is smaller than –2, although there is more of that quantity in –4.

As to the extension of addition and subtraction (and later multiplication and division), a great part of previous knowledge has to be accommodated. The part–part–whole structure relations stating that addition makes bigger, subtraction makes smaller (regarding the first number as a starting point), and the whole is bigger than each part no longer hold. For instance, in $(+7) + (–2) = (+5)$, +7 is transformed to +5, which is smaller, and in $(+2) – (–3) = +5$, +2 is transformed to +5, which is bigger. In other words, addition can make smaller and subtraction can make bigger.

The gap between naive and advanced conceptions of the part–part–whole structure, and the gap between naive and advanced conceptions of equality might be bridged by allowing the operation extension to be defined within algebraic situations, as demonstrated in the following example.

Let us consider again the On the Road problem, modifying it to have Anne traveling X km and then Y km northward (X or Y can be < 0, in which case the respective segment of the trip will be southward). The general expression for Ben's trip (to meet his wife), marked as S, would be: $S = X + Y$. When $Y = –60$, as in the original story, we get $S = X + (–60)$, and in the specific case $X = 40$ we get: $40 + (–60) = –20$.

Ben's trip, S, is mathematically equivalent to Anne's complete trip consisting of $X + Y$ because the net effect, in each case, is to displace a person

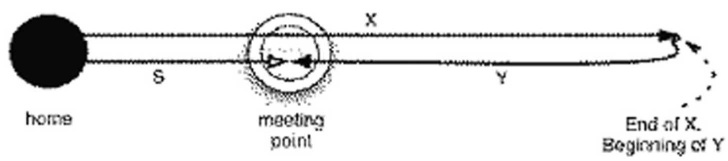

FIGURE 12.10. Equivalent routes I.

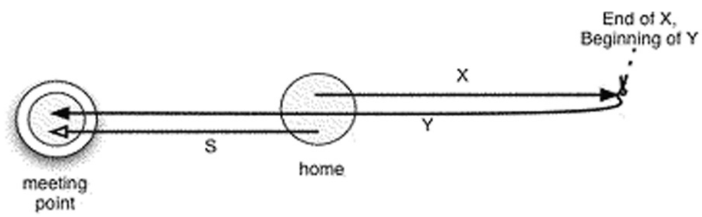

FIGURE 12.11. Equivalent routes II.

from home to the meeting point (see Figs. 12.10–12.12). However, Anne and Ben's trips are not conceptually equivalent when Anne's two segments represent different directions (see Figs. 12.10 and 12.11): Ben's trip is a shortcut. Because Ben proceeds directly to Anne at the end point of her trip, Ben's trip requires less driving, but, paradoxically, the net displacement is the same for Anne and Ben, thus we can now suggest a new extended meaning to the addition of signed numbers addition and subtraction.

Figures 12.10, 12.11, and 12.12 show some possible combinations of directions and sizes of the two journey parts. In all three examples, X is positive and rightward is northward, we can get symmetrical examples for $X < 0$ by reflecting the given examples (or regarding right as south). In the first two examples, Y is negative but has a different absolute value relative to X, and in the third case Y has the same direction as X.

In all the examples that can be generated for this situation, Ben's trip is expressed as the sum of the two parts of Anne's trip regardless of the signs of the parts, that is, $S = X + Y$. Similarly, we can express the second part of Anne's trip by looking at the difference between the whole (Ben's trip) and the part (Anne's) subtracting the part from the whole regardless of the number signs, that is, $Y = S - X$.

By offering this extension, we stretch the existing senses that children have about the operations and about equality. This bridge to an equivalence meaning of equation is different from the approach discussed by Filloy

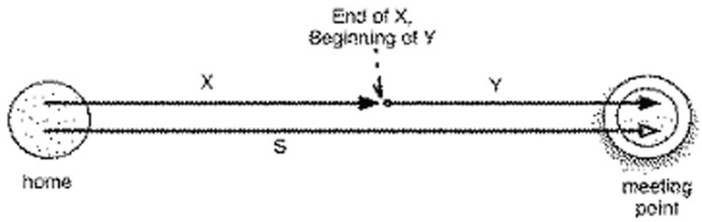

FIGURE 12.12. Equivalent routes III.

and Rojano (1989), who identify a didactical gap between arithmetic and algebra and view the transition to algebraic thinking as requiring the ability to operate on unknowns on both sides of an equation.

"The equivalence of routes" is a big idea because it requires suppressing otherwise important psychological features of situations in order to highlight a certain mathematical invariance. An analogous sort of reasoning is required when one deals with a string of transactions, each of which may correspond to a deposit or a withdrawal of funds in a bank account. One might sum for example 37 transactions over one week to determine a net effect, say, –$50 on one's bank balance. When a friend asks, "Has your bank account changed much in the past week?", one might answer, "Yes, I spent $50." This "short answer," as opposed to communicating the details of 37 transactions, corresponds to Ben's shortcuts in the first two trips.

Can young children understand the idea of equivalent transformations? The following section details an example of children's investigation of alternative routes and discusses its implications.

EQUIVALENT TRANSFORMATIONS: THE LONG WAY AND SHORTCUTS

The following episode from the Early Algebra Project (2000) demonstrated that third graders could discover shortcuts on the number line (i.e., a displacement that does the work of two consecutive displacements). At the same time, this episode exposes some difficulties in this process.

The Episode

Secret *start numbers* were disclosed to two children, each of whom performed two (known) displacements on the secret number and showed where they ended up on the number line. When Carolina ended at 3 and James at 5 after performing a +2 and then a –1 transformation, several children claimed (correctly) that they started at 2 and 4 correspondingly. At this point, Filipe suggested an interesting explanation:

> Filipe: I was just thinking it because you had a plus two and then you minused the one from the two, and then that was only plus one.
> Barbara: Ohh. Did you hear what Filipe said? He just gave us a shortcut. Did you hear Filipe's shortcut?

With the class-shared experience of "Filipe shortcut," the teacher assumes that the children understood the idea of finding one transformation as a shortcut alternative to two given transformations. The children, however, were more impressed with Filipe's role as the fairy godfather (in enacting one of the problems) than with his mathematical ideas. Thus, despite being exposed to the shortcut idea, several investigations were needed before more children understood it.

In the following task, the children were given a list of starting points. For each point they were expected to find where one would end up following two transformations: –1 and +4 and were also asked to think about any kinds [of] shortcuts you can find to do all of these problems. At first, children perceived the task as requiring the actual performance of the two moves (i.e., the ritual of moving by –1 and +4 was perceived by them as an integral part of finding $N - 1 + 4$, just as a young child sees the counting ritual as an integral part of answering the question, "How many are there?"). Performing the transformation in a different way still needed to be made legitimate. When the input number was 5 and Nathan wanted to give it a try, Barbara (the teacher) was expecting him to suggest a shortcut but, instead, he said:

> Nathan: Eight
> Barbara: How did you figure that out?
> Nathan: I did it on my ruler, cause five minus one equals four. Plus four. One . . . two . . . three . . . four . . . is eight.
> Barbara: Okay, . . . you did find a use for your ruler. You're using it like a number line, right? You're using all the numbers.

The next task involved pattern identification. Different input numbers were transformed by –5 and +4 and the children were asked to tell what they discover by looking at the input numbers, the corresponding output numbers, and the displacements. Most children could identify the –1 shortcut and the relevant connections. It should be noted that while doing the transformations, children had no trouble moving below zero. It was also interesting to note that some confusion was caused by zero as an input. Zero turned out to be a strange number to operate on, and involved –1 both as an output and a shortcut (i.e., as a point on the number line and as a transformation).

During several investigations of this nature, the children discovered the shortcut concept at different points along the sequence of tasks. Some discovered the repeating pattern but did not see the connection between the emerging shortcut and the two original transformations, that is, between the computation (–5) + (+4) = –1 and getting –1 as a shortcut in each specific case. Keeping in mind these were third graders, we can look at the glass half full and conclude that the equivalence concept is attainable, but the teaching trajectory should provide appropriate tasks.

CONCLUSIONS

In this chapter, we suggested that there is an interdependent relationship between algebra and signed number operations. In the first part, we argued that algebra provides a helpful context for introducing signed numbers. In the algebraic modeling of certain situations, one makes full use of operations defined on signed numbers and of the numbers as a combination of measure and direction. Arithmetical counterparts of such problems, on the other hand, do not require the full use of signed numbers. Thus, algebraic problems have the potential to facilitate the construction of a richer mathematical signed number operation model. Modified versions of problems such as the road problem and temperature problem can be used to generate teaching trajectories for introducing signed numbers and for introducing signed number operations. Similar problems can be later used to further enrich the mathematical model by applying it in a variety of situations where signed number operations genuinely contribute to the organization and analysis of these situations.

The second half of this chapter argued that signed number tasks can contribute to the understanding of algebraic concepts. The study of functions is more complete and meaningful with the extension of the number line to include negative numbers promoting the habit of checking different cases. Signed number tasks can help students move beyond the conception that equations display “an action on one side and its result on the other” to an “equivalence relation of transformations.” Similarly, signed number tasks can support the transition from arithmetical additive equations that have a part–part–whole structure to algebraic equations by providing an extended equivalence meaning for this structure.

REFERENCES

Arcavi, A. (1980). *Comparison of the effectiveness of four different approaches for teaching the multiplication of negative numbers.* Unpublished master’s thesis, Weizmann Institute of Science, Rehovot, Israel.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal*, *93*(4), 373–397.

Blanton, M., & Kaput, J. (2000). Generalizing and progressively formalizing in a third grade mathematics classroom: Conversations about even and odd numbers. In M. Fernández (Ed.), *Proceedings of the 22nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 115). Columbus, OH: ERIC Clearinghouse.

Carpenter, T. P., & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI study conference, pp. 155–162). Melbourne, Australia: University of Melbourne.

Carpenter, T. P., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades* (Research Report). Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science. Retrieved from http://www.wcer.wisc.edu/ncislapublications/index.html

Carraher, D. W., Schliemann, A. D., & Brizuela, B. (2000). *Children's early algebraic concepts*. Plenary address presented at the 22nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Tucson, AZ.

Carraher, D. W., Schielmann, A. D., & Brizuela, B. (2001). Can young students operate on unknowns. In M. V. D. Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 130–140). Utrecht, the Netherlands: Freudenthal Institute, Utrecht University.

Davis, R. B., & Maher, C. A. (1997). How students think: The role of representations. In L. D. English (Ed.), *Mathematical reasoning, analogies, metaphors, and images* (pp. 93–115). Hillsdale, NJ: Lawrence Erlbaum Associates.

Davydov, V. (1991). *Psychological abilities of primary school children in learning mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.

Freudenthal, H. (1983). Negative numbers and directed magnitudes. In H. Freudenthal (Ed.), *Didactical phenomenology of mathematical structures* (pp. 432–460). Dordrecht, the Netherlands: Reidel.

Gravemeijer, K. (1997). Solving word problems: A case of modelling? *Learning & Instruction*, *7*(4), 389–397.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York: Macmillan.

Greer, B. (1997). Modelling reality in the mathematics classroom: The case of word problems. *Learning & Instruction*, *7*(4), 293–307.

Janvier, C. (1985). Comparison of models aimed at teaching signed integers. In L. Streefland (Ed.), *Proceedings of the ninth international conference for the Psychology in Mathematics Education* (pp. 135–140). Utrecht, the Netherlands: Program Committee.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317–326.

Linchevski, L., & Williams, J. D. (1999). Using intuition from everyday life in "filling" the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics*, *39*, 131–147.

Mukhopadhyay, S., Resnick, L. B., & Schauble, L. (1990). Social sense-making in mathematics; Children's ideas of negative numbers. In G. Booker, P. Cobb, & T. N. de Mendicuti (Eds.), *Proceedings of the 14th international conference for the Psychology in Mathematics Education* (Vol. 3, pp. 281–288). Oaxtepec, Mexico: Conference Committee.

Peled, I. (1991). Levels of knowledge about negative numbers: Effects of age and ability. In F. Furinghetti (Ed.), *Proceedings of the 15th international conference for the Psychology in Mathematics Education* (Vol. 3, pp. 145–152). Assisi, Italy: Conference Committee.

Peled, I., Mukhopadhyay, S., &, Resnick, L. B. (1989). Formal and informal sources of mental models for negative numbers, In G. Vergnaud, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the 13th international conference for the Psychology of Mathematics Education* (Vol. 3, pp. 106–110). Paris, France: Conference Committee.

Rowell, D. W., & Norwood, K. S. (1999). Student-generated multiplication word problems. In O. Zaslavsky (Ed.), *Proceedings of the 23rd international conference for the Psychology in Mathematics Education* (Vol. 4, pp. 121–128). Haifa, Israel: Conference Committee.

Schliemann, A. D., Carraher, D. W., et al. (in press). *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice*. Mahwah, NJ: Lawrence Erlbaum Associates.

Shternberg, B., & Yerushalmy, M. (2003). Models of functions and models of situations: On design of a modelling based learning environment. In H. M. Doerr & R. Lesh (Eds.), *Beyond constructivism: A model and modelling perspective on teaching, learning, and problem solving in mathematics education* (pp. 479–500). Mahwah, NJ: Lawrence Erlbaum Associates.

TERC Tufts Early Algebra, Early Arithmetic Project. (2000, September). *Number line conversations about money among students in grade 3* (Classroom transcript). Cambridge, MA: TERC.

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education*, *19*(2), 115–133.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39–59). Hillsdale, NJ: Lawrence Erlbaum Associates.