TEN-YEAR-OLD STUDENTS SOLVING LINEAR EQUATIONS

BÁRBARA BRIZUELA, ANALÚCIA SCHLIEMANN [1]

We seek to re-conceptualize the perspective regarding students' difficulties with algebra. While acknowledging that students *do* have difficulties when learning algebra, we also argue that the generally espoused criteria for algebra as the ability to work with the syntactical rules for solving equations is rather narrow. Certainly, learning and understanding the syntactic rules for solving equations should be part of algebra instruction. However, algebra also involves work with other, multiple notation-systems (such as function tables, Cartesian coordinate graphs) and with variables and functions. Therefore, in our teaching experiments, it is only after the students have demonstrated familiarity with the concepts and representations of variables and functions, we introduce equations and work with algebraic syntax.

The crux of the argument of this article is that if we can present evidence of younger, elementary school children engaging with algebra, and using and understanding the syntactic rules of algebra, we have to ask ourselves why so many adolescents face difficulties with algebra. Perhaps it is not that the students are not prepared or ready for learning algebra, but that the teaching or curriculum to which the students have been exposed has been preventing them from developing mathematical ideas and representations they would otherwise be capable of developing.

It is our belief that, as previously stressed by many, the difficulties middle- and high-school students have with algebra result from their previous experiences with a mathematics curriculum that focuses exclusively on arithmetic procedures and computation rules. In this article, we describe classroom and interview data showing that, if children are given the opportunity to discuss algebraic relations and to develop algebraic notations, even ten-year-old students can solve algebraic equations with unknown amounts on both sides of the equals sign.

Background

Numerous researchers have suggested that algebra should pervade the mathematics curriculum and be introduced at an early age, thus avoiding an entrenchment of arithmetic (*e.g.*, Bodanskii, 1991; Booth, 1988; Brown and Coles, 2001; Crawford, 2001; Davis, 1985; Dougherty, 2003; Henry, 2001; Kaput and Blanton, 2001; Schoenfeld, 1995; Vergnaud, 1988; Warren, 2001). They agree that young students can grasp both algebraic concepts and notation, and that the boundaries between the arithmetic and algebra curricula can and should be downplayed.

Many researchers have suggested that students' difficulties with algebra arise from their previous lack of experience with arithmetic and algebraic generalizations that should be integral to the arithmetic curriculum:

> the difficulties that students experience in algebra are not so much difficulties in algebra itself as problems in arithmetic that remain uncorrected. (Booth, 1988, p. 29)

Given the problems in arithmetic, researchers have turned to introducing and embedding the elementary school curriculum with algebraic concepts. But, can children do it? We will illustrate this kind of work by describing how a group of ten-year-old students come to produce and solve linear equations with unknowns and variables on both sides of the equal sign, an achievement that has been thought to be out of reach for many middle- and high-school students.

There is relatively wide agreement among mathematics educators and among policy makers (see, for instance, NCTM, 2000; Sutherland, 2002) that algebra should become part of the elementary school curriculum. How this should be done, however, is still a matter of discussion and for systematic research. Successful implementation of algebraic activities and discussions in elementary school have been described by, for example, Bodanskii (1991); Carpenter and Franke (2001); Davis (1985); Dougherty (2003); Fujii and Stephens (2001); Kaput and Blanton (2001); and Schifter (1999). These researchers have aimed at algebra in the early grades, but their approaches have varied and focused on different aspects of what is considered to be algebra.

A first group of studies focused on the use of written equations and algebraic syntax, on children's understandings of equalities and inequalities, and on the use of letters to represent variables and unknowns:

> Bodanskii (1991), in Russia, focused mainly on the development and solution of written equations to solve verbal problems. He found that ten-year-old students who were introduced to algebra problems and notation for equations from first grade (six- or seven-year-olds) performed significantly better than twelve- and thirteen-year-old students who only had access to algebra from age eleven.
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> Brito Lima (Brito Lima and da Rocha Falcão, 1997) and Lins Lessa (unpublished doctoral dissertation) in Brazil, highlight issues related to both algebraic concepts and notation for equations. In their studies they have shown that elementary school children can develop written representations for algebraic problems and that their success on algebraic problems is based on the development of written equations and on the use of algebra syntactic rules for solving equations.
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> Carpenter and Franke (2001) focus on algebraic thinking through children's discussions on equations and inequalities. They show that young children, who participated in

classroom activities that explore mathematical relations, can understand and explain that an equation such as $a + b - b = a$ is true for any numbers a and b.

More recent experiences of introducing algebraic concepts from the early school years have been developed by the *Measure up* project in Hawai'i (Dougherty, 2003). The basic premise behind the project is the need to think in a non-traditional way about what constitutes appropriate mathematics for young children. What Dougherty and her colleagues propose, following Davydov's (1991) work, is for children to begin their mathematics instruction with no numbers. Instead, they focus on comparing physical attributes of objects. These comparisons and relationships between attributes are described with relational statements, using letters. Only gradually are numbers introduced into the curriculum, without ever setting aside the main focus on comparisons and relationships. In this project, children as young as five and six years old deal with equalities, inequalities, differences, commutativity, associativity, and inverses.

Davis's (1985) work in the Madison Project takes a different approach and focuses on eleven-year-old students working with concepts and notations for variables, Cartesian coordinates, and functions. Our own work has shown that nine-year-old students can learn to think of arithmetical operations as functions rather than merely as computations on particular numbers, that they can operate on unknowns (Carraher *et al.*, 2001) and work with mapping notation [2], such as $n \rightarrow 2n - 1$ (Carraher *et al.*, 2003). We have also found that function tables, graphs of linear functions, and establishing connections between tables and graphs are within reach of ten-year-old students (Brizuela, 2004; Schliemann and Carraher, 2002).

These demonstrations, however, may not have convinced some mathematics educators that young children can learn algebra. Previous research has highlighted students' *difficulty* in solving equations when unknown quantities appear on both sides of the equality (e.g. Filloy and Rojano, 1989; Herscovics and Linchevski, 1994). These researchers have attributed such findings to developmental constraints and the inherent abstractness of algebra, concluding that even adolescents may not be ready to learn algebra (Collis, 1975; Filloy and Rojano, 1989; Herscovics and Linchevski, 1994; Linchevski, 2001; MacGregor, 2001; Sfard and Linchevski, 1994).

Some researchers have claimed that students are engaging in algebra only if they can understand and use the syntax of algebra and solve equations with unknown amounts on both sides of the equals sign (see Filloy and Rojano, 1989). Filloy and Rojano (1989), for example, have proposed that there is a "cut-point" separating arithmetic from algebraic thought. Similarly, Herscovics and Linchevski (1994) point to "*students' inability to operate spontaneously with or on the unknown*" (p. 59, emphasis in the original) and refer to their failure to represent and manipulate unknowns as a cognitive gap.

Our approach to algebra notation

Algebra-symbolic notation is one of several basic representational systems of mathematics. In a *narrow* sense, algebraic reasoning is concerned only with algebra-symbolic notation. In the *broad* sense that we adopt in our research and in this paper, algebraic reasoning is *associated with* and *embedded* in many different representational systems. Although some educators argue against any and all uses of algebra-symbolic notation in the early grades, we feel it is better to frame the issue in a broad context. By *broad context* we mean to ask more generally how (different) written notations relate to mathematical reasoning and algebraic concepts in particular. Because algebra-symbolic notations are embedded and associated with algebra reasoning, and are integral to algebra, in our research we focus both on children's reasoning and on their use and understanding of algebra-symbolic notation.

Conventional notations help extend thinking (Cobb, 2000; Lerner and Sadovsky, 1994; Vygotsky, 1978), but if they are introduced without understanding, students may display premature formalization (Piaget, 1964). For these reasons, students need to be introduced to mathematical notations in ways that make sense to them.

Our approach relies on introducing new notations as variations on students' spontaneous ways of representing open-ended verbal problems. Our classroom intervention data has shown that young students can learn to use algebra-symbolic notation meaningfully to express generalizations they have reached while exploring problems in open-ended rich contexts. We have found that children can use mathematical notations not only to register what they understand, but also to structure and further their thinking, allowing them to make inferences they might otherwise not have made. We also showed that algebra notation can constitute a tool for generalizations, for understanding of linear functions, and for solving problems (Brizuela, 2004).

Our next step was to investigate whether elementary school children could also deal with written algebra equations and with the syntactic rules of algebra. Previous studies have shown that even seven-year-old students can understand a basic principle implicit in the rules for solving equations, namely, that adding or subtracting the same amount to two equal amounts does not destroy an equality, and that nine-year-old students can develop notations for algebraic problems and, with help from the interviewer, solve linear equation problems using different solution strategies, including the use of the syntactic rules of algebra (Bodanskii, 1991; Brito Lima and da Rocha Falcão, 1997).

In the longitudinal study we partially report here, we introduced children to equations as an extension of their work on functions, function tables, and graphs of linear functions. We will briefly describe the general approach we adopted for introducing children to algebra equations and will report on the final results of our study concerning ten-year-old students' use of algebra notation and strategies for solving equations. We will describe in detail:

- the students' discussions and written materials produced during the last lesson in one of the four classrooms we worked in, and

- the same students' responses to an individual interview.

The classroom intervention and its results

We worked with 70 students in four classrooms, from grades 2 (children between 7 and 8 years of age) to 4 (children between 9 and 10 years of age). Students were from a multi-ethnic community (75% Latino) in Greater Boston, and 83.09% of the students in the school were on the free or reduced meal program.

Each semester, from the beginning of their second semester in second grade to the end of their fourth grade, we implemented and documented six to eight Early Algebra activities in their classrooms, each one lasting about 90 minutes. The activities related to arithmetic operations, fractions, ratio, proportion, and negative numbers. Our goal was to examine how, as they participated in the activities, the students would work with variables, functions, positive and negative numbers, algebra notation, function tables, graphs, and equations.

The last six lessons we taught in fourth grade focused on equations and algebra notation. Each lesson focused on a problem that had unknown amounts in it and that could be represented with equations, as in the two examples shown in Figure 1.

> *Problem 1:*
> Mike and Robin each have some money. *Mike* has \$8 in his hand and the rest of his money is in his wallet. *Robin* has altogether exactly three times as much money as Mike has in his wallet. How much money could there be in Mike's wallet? Who has more money?
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> *Problem 2:*
> Which phone plan is better? *Plan #1:* You pay \$0.10 per minute for all calls. *Plan #2:* You pay \$0.60 per month plus \$0.05 per minute for calls.

Figure 1: The two examples.

When presented with the problems, children were not asked to find a 'right' answer, but to consider all possibilities, to draw the graphs of two functions, and to consider an answer only after they had gone through these steps. During the weeks leading up to the lesson we will focus on, the children felt fairly comfortable dealing with unknown amounts and some of the children were able to gradually use N to represent the unknown amounts, although some of them still used iconic notations. During the last lesson in fourth grade, the problem in Figure 2 was presented to the class.

> Two students have the same amount of candies. *Briana* has one box, two tubes, and seven loose candies. *Susan* has one box, one tube, and 20 loose candies. If each box has the same amount and each tube has the same amount, can you figure out how much each tube holds? What about each box?

Figure 2: The problem presented during the last lesson in fourth grade.

We chose this problem as it reflects part of the controversy regarding what exactly counts as algebra. This problem can be represented through the following equation:

$$y + 2x + 7 = y + x + 20.$$

The problem involves variables and unknowns on both sides of the equal sign, following Filloy and Rojano's (1989) constraints for when students are doing algebra. In this way we also adjust to Herscovics and Linchevski's (1994) constraint of learning algebra as involving manipulations of the unknown, which our students would have to do if they worked on the above problem.

We started off the lesson *acting out* the problem. At the front of the class, a box, two tubes, and seven candies in a transparent bag were put on a table that we labeled as Briana's table. Next to this table, a bag, a tube, and twenty candies in a transparent bag were put on a table that we labeled as Susan's table. Figure 3 shows the distribution of elements on the tables.

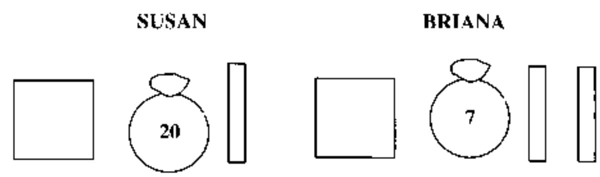

Figure 3: The distribution of elements on the table for Susan and Briana's candy problem.

In our work with the children, we had encouraged them to use two different kinds of strategies when dealing with algebraic problems. The first had to do with 'matching up' amounts on both sides of an equal sign, or matching up amounts belonging to two different people, such as Susan and Briana. The second strategy had to do with canceling out equal amounts on both sides of an equal sign or amounts belonging to two different people. Using these different strategies, we can think of at least two different approaches to dealing with the problem at hand. Figure 4 shows one possible approach.

In this approach, you cancel or match up Susan and Briana's boxes as well as Susan's tube of candies and one of Briana's tubes. At this point, we are left with both girls' loose candies and one of Briana's tubes. Assuming that both girls have the same amount of candies, we think of the amount of candies in Susan's bag as two sub-amounts: the

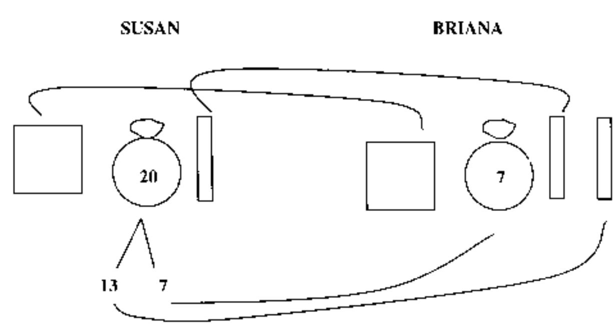

Figure 4: One potential approach to the problem.

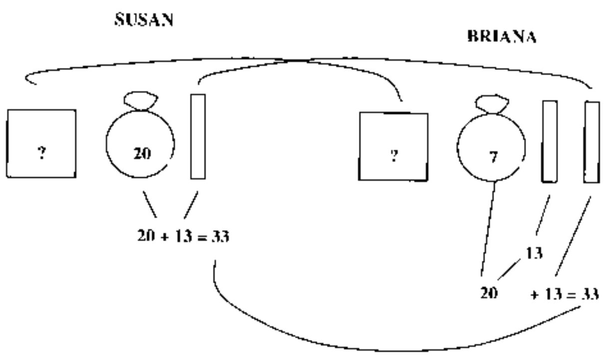

Figure 5: Another potential approach to the problem.

amount that corresponds to Briana's loose candies (7) and what is left of Briana's candies – the tube. Given that 20 – 7 = 13, we can assume that Briana's leftover tube has 13 candies.

Figure 5 shows another potential approach to the problem. In this approach, the two boxes of candies are matched up, as well as one of each of the girls' tubes of candies. Then, again assuming that both girls have the same amount of candies, we can assume that Susan's loose candies – 20 – are equal to Briana's loose candies – 7 – and her leftover tube, thus $7 + x = 20$ and the amount in the tube must be 13 candies. These are only two potential approaches. We envisioned children adopting one or the other approach, or a combination of both, or some original way of dealing with the problem of figuring out how many candies were contained in the tubes.

The researcher teaching this particular group of students (David) generated a teaching and learning environment that was conducive to children's presentation of their own perspectives, ideas, and ways of representing problems. An open-ended word problem was usually presented to the children at the beginning of each ninety-minute lesson.

First, children's reactions to the problem were brainstormed. Then, children were asked to show, on paper, their ideas about the problem and their suggested solutions, if these were asked for. These notations were then shared by the children at the front of the class. Usually, this involved David selecting some representative examples to present to the whole class. These group presentations then led to some more group discussion about the problem and to some sort of consensus about the problem and its solutions. David was the teacher for this group of students only during our Early Algebra interventions. At other times, the children had a classroom teacher who was a participant observer to our lessons. Our lessons responded to the needs presented by the teacher, as well as the state curriculum frameworks that the children's curriculum was designed around.

Given the scenario at the front of the class, the students started by discussing the problem. Aarielle recalled that this problem was similar to the "wallet problem" (see above) they had solved six weeks before. Kauthamy stated that Susan had thirteen more candies in her bag than Briana did in hers, and Albert observed that Briana had an extra tube of candy. Thus, both Kauthamy and Albert started out by highlighting the differences between Susan and Briana's amounts; Susan has 13 more loose candies, and Briana has an extra tube of candies. The combination of these initial observations will prove key in the children's dealing with the problem.

When David asked if they could figure out how many candies there were in a tube or in a box, most of the students answered that they could not. However, less than fourteen minutes into the class, Albert continued to elaborate on his initial idea and he explained that Briana's tubes had to have thirteen candies in them so that one of her tubes plus the seven loose candies could be equal to Susan's twenty loose candies! At once, other children in the class agreed with Albert.

Up to this moment, the children in the class focused, spontaneously, on the amount of candies contained in the tubes. Thus, they were focusing on the unknown amounts on both sides of the equal sign, as defined by an algebraic expression of the problem. At this point, however, Cristian noted that it did not matter how many candies were in the boxes, thus pointing, for the first time, to the variable nature of the amount of candies in the boxes. The amount of candies contained in the boxes was variable, and could be given any possible value as long as both boxes each contained the same number of candies. Cristian also explained that if Briana's tube had 13 candies, then these candies added to the 7 loose candies would make it 20, plus 13 more for her second tube would be 33. For Susan, she has 13 in her tube plus the 20 loose candies, which is 33. He said that because of this, the two girls' total amounts are equal no matter how many candies are contained in the boxes. Figure 6 shows Cristian's thinking in a diagrammatic way.

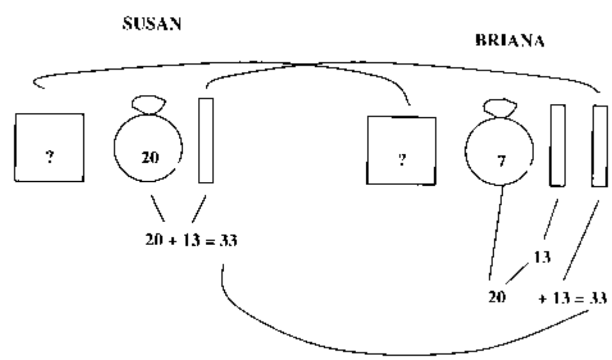

Figure 6: Cristian's thinking about the problem.

Mariah, however, unsure of Albert's reasoning, asked him to explain why he thought there were thirteen candies in each tube. He answered that the amount in a tube plus the seven loose candies that Briana had, had to be equal to Susan's twenty loose candies. Mariah then asked about Briana's second tube and Albert assured her that his proposal would still work because Susan also had one tube. Figure 7 shows Albert's thinking in a diagrammatic way.

Carissa further explained that the candies in Susan's bag – the loose candies – made up for the extra tube that Briana had. A little later, Carissa explained that Briana and Susan had the same amount of candies and thus Susan's bag, that had twenty loose candies, was really like thirteen plus seven. She also explained that, when comparing the loose candies

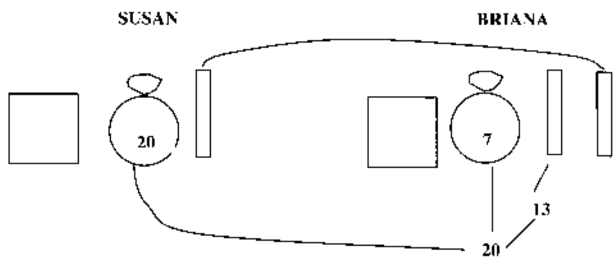

Figure 7: Albert's thinking about the problem.

that each one of the girls had, Susan ended up having thirteen extra candies; those thirteen extra candies that Susan had matched up exactly with the amount in Briana's extra tube. Thus, Susan's extra candies when compared with Briana's loose candies –13 – could be placed in an extra tube for her, making Briana and Susan each have two tubes of candies, one box, and 7 loose candies. Figure 8 shows Carissa's thinking in a diagrammatic way.

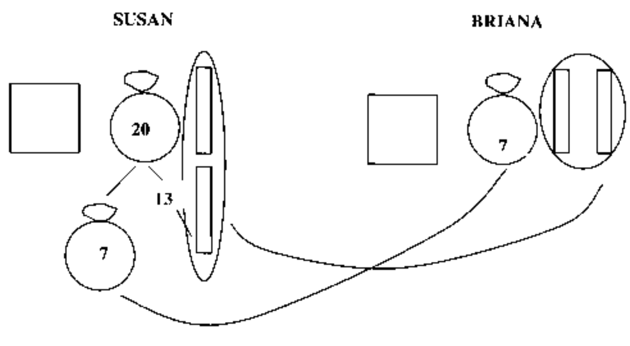

Figure 8: Carissa's thinking about the problem.

David (the instructor) asked how many candies in Susan's bag (the loose candies) made up for Briana's extra tube and Albert replied thirteen, which would leave seven extra candies (the amount of loose candies Susan had). A few minutes later, David asked, "How do we know that the tubes have thirteen and that the girls are holding the same amount if we haven't peeked in the tubes yet?" Cristian replied that this is called algebra and Briana and Mariah explained that they used algebra to subtract and make educated guesses. David challenged the children to prove that there were indeed thirteen candies in a tube. Cristian explained that we could use *N* to stand for a tube, and the class as a whole agreed that a different letter should be used to stand for the boxes.

Subsequently, the students sat down to work on ways of representing the problem in writing. Each of the students in the class produced their written account of the problem. Although most of the children in this class of eighteen students made iconic notations for the problem (78%), one third of them included an equation in their notation and more than one third (39%) included a letter to stand for one or more of the unknown values.

Nancy's written work (see Figure 9) is an example of an iconic notation. She first worked with the amounts given for the loose candies (twenty and seven) and correctly used the difference of thirteen between these two amounts as the value for what was inside the tubes, showing one tube on Susan's table and both of Briana's tubes as having thirteen in them.

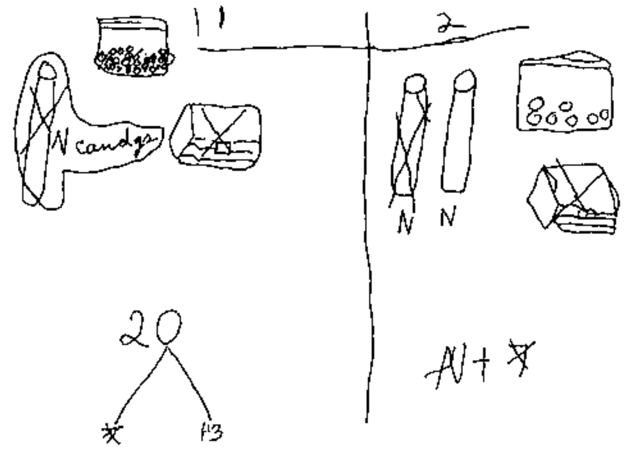

Figure 9: Nancy.

Although Nancy acknowledged that Susan started out with twenty loose candies, on the table she showed her as having seven and thirteen – just like Briana. One interesting feature of Nancy's work is the question mark that she placed on the two boxes – the amount of candies in the boxes was unknown (hence the question marks) and would and could remain unknown to the very end of the problem. In our longitudinal study, this was not the first time that we had observed children using question marks to represent unknown amounts.

Ramón's written work (see Figure 10) is also interesting in the way in which he was able to integrate both iconic and algebra notations (N + 7). He consistently used the *N* to

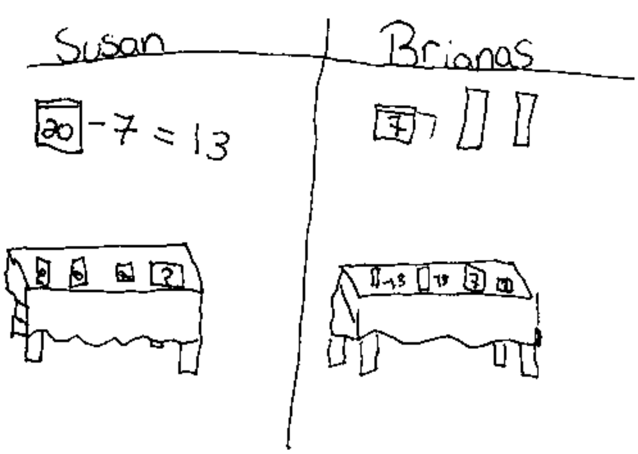

Figure 10: Ramón.

show what was in the tubes of candies. In addition, he used his notation to solve the problem and show his solution to the problem. He represented Susan's (1) and Briana's (2) candies iconically, then matched what they had and crossed out the matching amounts on both sides. He did not assign a value for the boxes and appeared to have no problem crossing them out since there was one on each side of his notation. Through his matching, he arrived at the conclusion that, in order to have equal amounts of candy, Susan must be left with seven and thirteen candies (totaling twenty candies) and Briana must be left with $N + 7$, making N be thirteen candies.

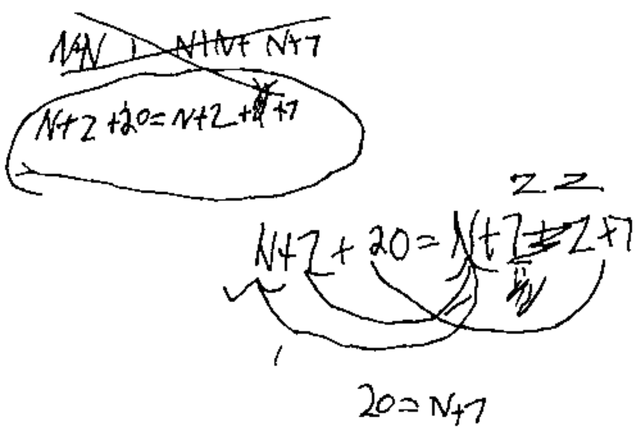

Figure 11: Albert.

Figure 11 shows that Albert used an equation to represent the problem. He used both N and Z as the unknowns. He started by using N to represent the amounts in the tubes and in the boxes but soon used Z for the amount of candy in the tubes. After matching the equal amounts, he appeared to have used the letters interchangeably as he finally reached the equation $20 = N + 7$.

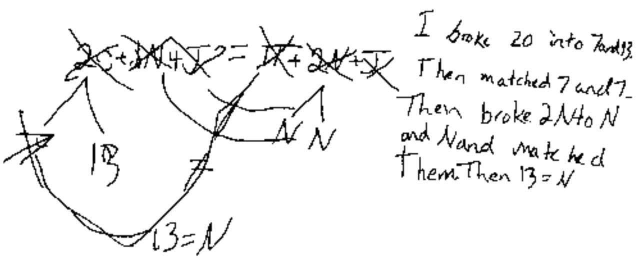

Figure 12: Cristian.

Figure 12 shows a very sophisticated notation by Cristian. Although similar to Albert's, Cristian's notations are of added relevance given the explanation written out at the side of the equations that he matched up:

> I broke 20 into 7 and 13. Then matched 7 and 7. Then broke $2N$ to N and N and matched them. Then $13 = N$.

Cristian set Susan's and Briana's amounts equal and matched the elements on both sides of the equation.

Once the group as a whole met to discuss the problem and the students' written notations, David wrote on the board that T is the amount in each tube and B is the amount in each box. He then asked the class, "How much did Briana have?" The class confidently called out:

$2T + B + 7$.

He then asked, "How much did Susan have?" and the class called out:

$T + B + 20$.

When David asked whether these expressions could be simplified, Albert suggested matching up the Bs. David did so and crossed them out, saying:

> Now we have $2T + 7$ and on the other side we have $T + 20$. How could we simplify them further?

Carissa, turning to the scenario shown on the tables at the front of the class, suggested putting seven candies in Susan's bag and leaving thirteen out in a pretend tube. David did so and Aarielle wrote on the board, breaking up the twenty into thirteen and seven and matching up the two sevens. David erased the 7 from the board to leave $2T = T + 13$. Cristian suggested matching two tubes to leave $T = 13$. David did this with the actual tubes and recorded it on the board: "So T has to be 13. Let's count the tube candies." Subsequently, Kauthamy counted the candies and found thirteen. The children shouted out, "Hooray!", expressing their excitement at their solution to the problem.

Interview results

At the end of the school year, one to four weeks after the last class, we individually interviewed the children on a series of problems. In the last part of the interview, children were asked to represent in writing and to solve the problem shown in Figure 13.

Harold has some money. Sally has four times as much money as Harold. Harold earns $18.00 more dollars. Now he has the same amount as Sally. Can you figure out how much money Harold has altogether? What about Sally?

Figure 13: The interview problem.

Of the eighteen children from this class who were interviewed, ten represented Harold's initial amount as N, X or H, and Sally's amount as $N \times 4$. For Harold's amount after earning 18 more dollars, eight children wrote $N + 18$. Four children wrote the full equation $N + 18 = N \times 4$ and eight children correctly solved the problem. However, only one systematically used the method of matching up equivalent terms on both sides of the equal sign to simplify the equation. Another child, when prompted, correctly explained this method. Apparently, as the children worked in their written notations, they easily inferred that Harold's starting amount was 6. As Albert stated, "I thought about six because it just popped in my head."

Discussion

The lesson, and children's reactions to the problem presented, suggest a number of issues related to algebra in the mathematics curriculum. First of all, we have presented evidence of young, elementary-school children doing algebra. The kind of problem they engaged with could be represented by an expression that included unknown and variable amounts on both sides of the equal sign, thus satisfying the constraints set for a particular vision of algebra as involving the use of algebraic syntax.

One third of the children used an algebraic expression to represent the problem, and over a third of the children included a letter to stand for one or more of the unknown values. Thus, we return to the question posed at the beginning of this paper: if such young children can succeed at working and understanding algebra, why is it that so many adolescents have trouble with algebra?

The children we were working with had been exposed to from six to eight early algebra activities each semester for the previous year and a half. Thus, we could argue that it was this exposure that enabled them to approach the problem successfully, to use algebraic expressions, and letters in their notations. However, this still leaves two issues exposed: the students are young, and doing at most 24 activities over the course of a year and a half is not significant.

So we come back to the same argument presented at the beginning of this paper: the location of the problem cannot be found in the students themselves. If younger students were able to work with algebra, older students could potentially as well. It is most likely the type of instruction that the students are exposed to that is impacting on their performance in algebra. If students were exposed to an algebrafied mathematics curriculum from the onset of their education, it is likely that by the time they are adolescents, they would be able to deal with a lot more complex mathematics.

The kinds of activities we developed over the last six weeks of our longitudinal study were not simple or easy for the students. Nevertheless, they were able to deal with the challenges we proposed and, at the end of only six meetings on equations, many were able to represent, discuss meaningfully and analyze problems involving unknown amounts on both sides of an equality. In the classroom, at least a third of the students in this class could represent the problem as an equation, solve the equation, and meaningfully explain why they could manipulate the elements in the equation. In the interviews, more than half of the children correctly represented the amounts in the problem using letters to stand for unknown amounts.

Our results suggest that dealing with equations is not beyond ten-year-old students' mathematical understanding and that much more could be achieved if the same kind of activities become part of the daily mathematics classes offered to elementary school children.

Notes

[1] This article is part of a larger longitudinal study sponsored by the National Science Foundation (Grant #9909591, awarded to D. Carraher and A. Schliemann, see earlyalgebra.terc.edu).

[2] We use the term notation to refer to written, external representations (see Lee and Karmiloff-Smith, 1996, who also state that these notations belong to notational systems with which they share a set of rules).

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But what about "me"? Suppose I am a blind man, and I use a stick. I go tap, tap, tap. Where do *I* start? Is my mental system bounded at the handle of the stick? Is it bounded by my skin? Does it start halfway up the stick? Does it start at the tip of the stick? But these are nonsense questions. The stick is a pathway along which transforms of difference are being transmitted. The way to delineate the system is to draw the limiting line in such a way that you do not cut any of these pathways in ways which leave things inexplicable. If what you are trying to explain is a given piece of behaviour, such as the locomotion of the blind man, then, for this purpose, you will need the street, the stick, the man; the street, the stick, and so on, round and round.

But when the blind man sits down to eat his lunch, his stick and its messages will no longer be relevant – if it is his eating that you want to understand.

(Bateson, G. (2000, first edition 1972) 'Form, substance and difference', in *Steps to an ecology of mind,* Chicago and London, The University of Chicago Press, p. 465.)