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## MODELING REASONING

**Abstract.** The three chapters in this section approach modeling from very different yet consistently intriguing points of view. Nemirovsky focuses on how people generalize; he argues that generalizations are intimately grounded in situations and only exceptionally take the form of “For all  $x$ , the following holds...” He illustrates his view through a detailed analysis of how a student creates representations to model the passage of a train through a tunnel. Verschaffel, Greer, and De Corte review their program of research about how students and teachers interpret mathematical word problems. They argue that classroom cultures tend to encourage students to ignore real-world constraints and considerations crucial to problem solving in out of school settings. Further, they claim that for mathematics to be useful in modeling problems of a practical nature it is important that mathematics education give due regard to realistic considerations in modeling. Kaput and Shaffer discuss human reasoning, including modeling, from an evolutionary perspective. They adopt Donald’s theory that human representational competence has gone through stages marked by radically new means of using symbols. And they propose that human cognition is presently moving toward a fifth stage, where computer technology augments people’s ability to represent scientific and mathematical problems.

### 1. GENERALIZING WITHOUT MAKING A GENERALIZATION

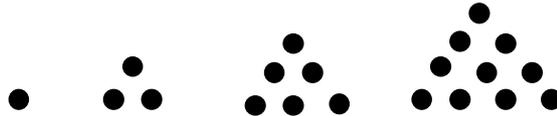
Nemirovsky’s chapter focuses on the nature of mathematical generalization. First he points out that some forms of generalization are dismissive in nature and provide an impoverished (“flattened”) rather than enriched understanding of phenomena. Next, moving to an ethical context, he compares Plato and Aristotle on the nature of concepts. He asserts that their views can be

captured...by a distinction between formal and situated generalizations. Formal generalizing involves a movement away from the circumstances; it is an effort to find an essence that can be separated and independently defined, apart from the noisy surroundings of the events that are being generalized. Situated generalizing is a movement toward the circumstances. (Nemirovsky, this book, p. 236)

These ideas set the stage for the rest of the chapter, in which he deepens his analysis of situated generalizing and argues that it offers a better model of how people actually reason than formal generalizing does.

In mathematics education, it is often very difficult to identify cases of generalization. Consider a case where students begin to notice a pattern but do not find words to adequately express it. For example, they may correctly extend a series of figures (see Figure 1) before being able to explain how to determine the number of dots for *any* instance. The fact that they recognize and can extend a pattern shows some degree of generalization. But they have not ‘made a generalization’ in the traditional, formal sense. We would say that they have generalized but have not

expressed this generalization in a linguistic form, much less in a conventional mathematical form (such as recursive equations with one unknown).



*Figure 1: A sequence of triangles.*

We came across other curious examples of ‘knowing something general without making generalizations’ in our work with street vendors in Brazil. In learning to handle simple commercial transactions—selling fruits and vegetables—street sellers revealed that they had acquired a general understanding of the structure of our number system. For example, they implicitly understood the associative property of addition and subtraction: quantities could be broken into sums of component quantities, operations could be carried out on the components, and the results could be joined in order to arrive at final answers consistent with the information given. The street sellers’ knowledge of the associative property was tacit. We might use the expression ‘theorem-in-action’ (Vergnaud, 1994) to account for this tacit knowledge or ‘knowing through doing’. Compared to a mathematical theorem, a theorem-in-action is restricted and implicit. Nonetheless, it has the potential to become explicit. Arguably, much of a mathematics educator’s work consists in helping students come to articulate, formalize, and coherently systematize their informal understandings. This bears striking similarity to the transition from the logic of discovery to the logic of justification and proof in the work of professional mathematicians and scientists.

We coined the expression ‘situated generalization’ several years ago (Carraher, Nemirovsky, & Schliemann, 1995) partly to capture this tacit form of generalizing. However, there was another, related issue we wished to highlight; namely, reasoning could both be closely linked to particular features of the current situation while exhibiting properties of a general, non-situation-bound character.

In analyzing video episodes with Nemirovsky and his colleagues (including those examined in Nemirovsky’s chapter) we became intrigued at how students came in touch with general issues while trying to solve local, apparently situation bound problems. In order to use a distance-time graph to solve a particular problem—predicting the region of a tunnel where a train was located—Clio drew upon arithmetical and spatial knowledge of a general nature. In a sense ‘situated generalization’ expressed our hopes that a situated learning approach could be reconciled with the belief that there exists knowledge of a trans-situational character. Likewise it expressed our distaste for the abstract-concrete opposition that often obscured more than it illuminated. In his chapter, Nemirovsky develops the concept more fully and has imbued it with his own insights on generalization.

The idea of ‘noticing resemblances’, inspired by Wittgenstein’s ‘family resemblances’, is key to Nemirovsky’s notion of situated generalizing:

The accrual of life experience is what enables us to notice certain resemblances and not others. By 'noticing resemblances' I mean recognizing likeness, such as, for instance, perceiving that two faces are similar. (Nemirovsky, this book, p. 238)

*Noticing resemblances* merits careful consideration because 'likeness' (or 'similarity') lies at the heart of generalization and proves especially problematic for mathematical concepts.

Wittgenstein's notion of *family resemblances* appeals to the fact that family members may resemble each other even though they share no common trait among them such as aquiline noses or brown eyes. Although at first glance Wittgenstein appears to address merely physical likeness, this is not the case. In fact he (Wittgenstein, 1987/1953) proposes the idea of family resemblances to deal with the concept of *game*. He understands that athletic games, board games, games of chance, language games and so on share no common elements and yet are all legitimate examples of games. In other words, the abstract concept of game is not based upon a sharing of common elements.

This is precisely the point Cassirer (1953/1923) made in his critical analysis of Aristotle's view of concepts. If general concepts (e.g. number) derived their meaning from the elements shared by subordinate concepts (e.g. integer, rational number, algebraic number, real number...) then these general, more abstract concepts would tend to be vague and pallid. Cassirer proposes that a general concept subsume the subordinate cases in all their detail. The mathematical concept of *number* could thus embrace the particular qualities of each subclass of number. This allows abstract concepts to be imbued with rich detail and subtle nuances, thereby including concrete features and instances.

Nemirovsky, like Cassirer, treats concrete and abstract as intertwined. Whereas Cassirer's claim arises from epistemology, Nemirovsky's point is more psychological in nature and takes concrete, lived experience as the point of departure: concrete circumstances are imbued with general meaning because of the experience people bring to bear on particular events. If one really wants to understand human reasoning, he argues, we need to consider the particulars of lived situations, for therein lies the general. In proceeding thusly, certain features emerge that no theory of human reasoning could predict. For example, he notes how Clio attaches Post-It notes onto the position-time graph to represent the tunnel sections and thereby understand better how the tunnel configuration relates to the regions and dimensions of the graph space. Such observations suggest that insight is a gradually unfolding process involving a fair degree of opportunism on the part of the problem solver (and the interviewer!). And they lend support to Nemirovsky's view that reasoning cannot be adequately explained by appealing to the existence of mental structures that, some would argue, predetermine the outcome of problem solving. Some of the issues Clio is facing are peculiar to the circumstances and representational system at hand. Clio never represents the problem in a general, situation-free form.

So much for the *situated* quality of her reasoning. What is *general* about Clio's solution?

For Nemirovsky:

Clio has developed, and is further refining, a general method to locate the train under the tunnel from the computer graph; it is general in the sense that it can be used for any box location of the train and it is a generalization because she used a few basic measurements to work for all. Clio's approach does not fit the standard form of a generalization statement (i.e. 'all x are y') but this shows an important point, which is that generalization is not a matter of making isolated statements. (Nemirovsky, this book, p. 250)

If Clio analyses the graph on the computer screen, she can describe the trip the train took, and furthermore predict the tunnel under which the train has stopped, even though she did not watch the train as it moved through the tunnel. One suspects that she could do this for nearly *any* such graph in those circumstances.

Clio understands the causal relations linking the two environments and is capable of mentally structuring how locations and movements in one system correspond to locations and movements in the other, despite their enormous differences (read "lack of physical resemblance"). By considering these general aspects of Clio's knowledge, her ability to work with causal relations, her ability to make systematic relationships from one spatial representation to another, one gains a sense of the deliberateness and awareness that lie behind her ways to deal with the problem at hand.

One may still wonder how Clio's story relates to mathematics education. Where does the mathematics lie? Does Clio demonstrate mathematical learning, mathematical generalization? Or just generalization?

Clio's interview relates to the concept of mathematical functions. Clio is not thinking about functions as general mathematical objects. The data are not arbitrary mappings of points to points. Clio knows that the train moves forward or backward at a speed that depends upon where she positions the controller lever, and that the graph on the monitor emerges somehow as a result of the train's movement. Because she understands these causal interrelationships she can look to the graph as a record of the train's movement. She cannot immediately 'read' the graph but must gradually develop an understanding of how specific events and structures occurring in the interviewing room play a role in the behavior and shape of the computer graph. She has already learned a lot about how the motion detector system works from two previous sessions. She learned that her own movements toward the receiver caused the graph to move toward the bottom of the screen whereas movements away from the receiver caused the graph to move upwards. She also came to realize how to associate slopes of the graph with the speed and direction of her movements.

The fact that her understanding does not originate through a direct contact with mathematical objects should not diminish the importance of abstraction and formalization in mathematical thinking. Perhaps we need to suppress the urge to view formal and situated approaches as mutually exclusive alternatives. Although mathematical knowledge originates from acting and reflecting on actions and properties of particular situations, it is characterized also by the gradual creation of overarching concepts and the appropriation of general representational tools that allow one to construct mathematical objects that cannot be reduced to any particular

experience, to any particular embodiment. Formalization is an important kind of generalization, despite the fact that it is so often abused in mathematics education. When students meaningfully express ideas in the formal language(s) of mathematics, they transform their knowledge, extending its reach and meaning. Their formal representations can serve as objects of reflection and inquiry, thereby playing a role in the future evolution of their mathematical understanding.

## 2. MATHEMATICAL MODELS AND REALISM

Whereas Nemirovsky takes a philosophical approach to modeling, and Kaput and Shaffer take a historical and semiotic one, Verschaffel, Greer and De Corte look at modeling as it typically occurs in mathematics classrooms—that is, modeling based on word problems. They work from the premise that modeling, which they describe as the “application of mathematics to solve problem situations in the real world”, is essentially about producing practical and useful solutions.

Several years ago investigations from a variety of studies presented compelling evidence of ‘suspension of sense making’ by students when solving certain types of word problems. For example, a student might determine that each child at a birthday party should receive 4.5 balloons. Or that an athlete will run the same average speed over one mile that she runs in a 100 meter dash? Or that water entering at constant rate into a sloped flask will rise at a constant rate.

Such errors seem to arise for reasons other than alertness, because, as the authors learned from their subsequent work, even when children are warned that problems may be tricky or may be impossible to solve, they tend to commit errors at about the same rate. It helps, however, if students are placed in a situation where they must make a practical decision on the basis of their conclusion (e.g. calling a rental company on a telephone to order a certain number of vehicles). But their research also shows that teachers themselves respond to ‘problematic’ word problems in ways similar to students.

The authors gradually began to interpret such results in terms of tacit assumptions adopted in mathematics instruction. They note that educators and curriculum developers routinely devise problems that remove any need for students to think about the diverse possible ways to represent the problem and judge among alternatives.

The ‘rules of the game’ that the authors identify provide thought-provoking ideas about why students and teachers will furnish seemingly outlandish answers to simple arithmetic problems. For example the authors mention the following ‘make believe’ assumption adopted in mathematics classes:

Assume that persons, objects, places, plots, etc., are different in a school word problem than in a real-world situation, and don’t worry (too much) if your knowledge or intuitions about the everyday world are violated in the situation described in the problem situation. (Verschaffel, Greer & De Corte, this book, p. 265)

In fact, the authors make such a compelling case for realistic modeling that we might wonder why educators might ever be willing to work under such artificial premises. Imagine that an instructor is covering issues of ratio and proportion in a

fifth grade lesson and introduces a commercial transaction as a context for dealing with relations of direct proportion. She might tell students that two dozen eggs cost one dollar and ask them to determine how much one would have to pay for 36 eggs, 600 eggs, or 6000 eggs. One would normally get a discount when purchasing large amounts of a product. The rationale for ignoring this possibility clearly lies in the educator's hope that the problem will provide the students with the opportunity for thinking about proportional relations.

Guberman (1998) notes the case of a teacher who gave students the task of determining the cost of a Thanksgiving dinner. One of the groups 'solved' the problem by looking through the advertisements in a newspaper and selecting a restaurant with an advertised price for a turkey dinner. In a sense, the students may have met the real world constraints of the problem, particularly if they were price-conscious in their selection. And perhaps they should have been commended for their creative circumvention of the 'rules of the game.' But the teacher will probably be unhappy with their answer, not necessarily because she disapproves of them dining out on Thanksgiving, but rather because the students have managed to answer the problem without giving any thought to the mathematical issues involved in determining the cost of a meal.

We suspect that as students advance in their mathematical careers, it becomes increasingly difficult to find realistic problems to illustrate the mathematical ideas. In probability theory one works with an idealized dice: we assume the likelihood of throwing a one, a six, or any number between them, is exactly  $1/6$  even though we know that such dice do not exist in the real world. We eliminate all but the most circumscribed and logical of human activities, such as board games, to mathematically represent choice situations. We know that people consider a wide range of factors when taking out a loan for their home (their trust in the banker, the size of the monthly payment, the convenience of using a bank where they already have accounts). But if these issues were given weight in a class about financial statistics, it would be hard to give proper attention to inherently mathematical issues, such as comparing the interest to principal ratios for loans of different durations.

In fairness, the authors realize that there needs to be a balance between the emphasis upon the utility of mathematics on one hand, and on the mathematical structures, on the other.

We acknowledge that our plea for a genuine modeling perspective does not exclude that word problems have different roles in elementary mathematics education. At one time they may be used mainly to create strong links between mathematical operations and prototypically "clean" model situations (with little room for endless discussions about the situational complexities that might jeopardize this link). But at other times they may be used primarily as exercises in relating real-world situations to mathematical models, in comparing the merits and pitfalls of different alternative models, and in reflecting upon that complex relationship between reality and mathematics. (Verschaffel, Greer & De Corte, this book, p.. 273)

This search for a proper balance reminds us of our own struggle to reconcile the relationship between everyday and academic mathematics (Carraher & Schliemann, in press). Since situations never map perfectly onto mathematics, there will always be a tension between formal considerations and practical concerns. Rather than favor

one approach invariably over the other, one hopes that students could handle a range of approaches to any given problem and even be able to compare and contrast them in terms of the virtues they embody.

### 3. TECHNOLOGIES AND COGNITION

Kaput and Shaffer take a broad, sweeping look at the nature of human symbolic competence, adopting and suggesting an extension to Donald's (1991) evolutionary analysis. Donald's model holds that human representation evolved over millions of years from an initial stage of *episodic thought* (thinking based on literal recall, without language, of events), followed many millennia later by a gestural, *mimetic representation* (think of *mime*, as in pantomime), characterized by "conscious, self-initiated, representational acts that are intentional but not linguistic." In a third stage, begun about 300,000 years ago, human thought and communication became organized around spoken languages with a proper syntax, advances associated with the emergence of mythic cultures. And in the fourth stage, written symbols and paradigmatic thought undergird a "theoretic culture" with the emergence of science and specialized professions increasingly organized around written records and external, conventional symbol systems.

The authors highlight how human representational competence developed not only as a result of built-in biological advances, but also as a result of cultural developments—a theme redolent of LaBarre's (1961) analysis. Such ideas led Donald to describe present-day humans as having a *hybrid mind*, constituted by contributions from biological and cultural endowments.

Of particular interest are the authors' remarks on the gradual changes in symbol systems over long periods of time. They note for example that iterative token systems gave rise to slotting systems (e.g. reckoning tables) in trade and accounting and ultimately to algorithms, in which operations are carried out on and through linear script. Surely these cultural inventions and refinements have ramifications for how mathematics is done; what effects, if any, do they have on the minds of their users?

Kaput and Shaffer argue that culture, and the individuals within it, are now moving into a fifth stage characterized by an increasing tendency to represent problems and think with computer assistance. The power of new information technologies lies in their helping us easily achieve what would otherwise take enormous effort.

The chapter implicitly hovers around the notion of cultural amplifiers—artifacts, physical or conceptual, that help people accomplish certain tasks. Just as a hammer augments the amount of force that a human hand can apply to a nail, arithmetical algorithms and algebraic systems greatly increase people's ability to solve certain classes of mathematical problems, as Kaput and Shaffer aptly note:

We can type the following two-variable function into a computer and see the surface that constitutes its graph, as in Figure 2, within a fraction of a second:

$$z = [\sin xy + \_ \cos 2x + 1/3 \sin 3y + \_ \cos 4(x+y)] / [1 + \sin 5y + \_ \cos 6x + 1/3 \sin 7y + \_ \cos 8x]$$

Moreover, we can then use the mouse to manipulate that graph as if it were a physical object—turn it on its side, rotate it, etc. Even more significantly, any constant in the function can be treated as a parameter and allowed to range over whatever domain we choose to define. In other words, this can be experienced as a class of functions, not a single function. (Kaput & Shaffer, this book, p. 286)

Representational and computational tools frequently serve us by taking over tasks and handling issues that we would otherwise be required to solve by ourselves. In the graphing example, software users do not have to concern themselves with point to point plotting, with generating three dimensional images of a two dimensional surface, and so on.

That people can accomplish more with such artifacts is not under dispute. However many people wonder whether cognitive processes themselves undergo fundamental changes by virtue of such an alliance (Cole & Griffin, 1980).

Vygotsky noted that tools, artifacts, and cultural representations not only introduce new cognitive functions connected to their use, but also diminish the use of “natural processes”, replacing and reorganizing certain mental functions. Thus some representations and tools may render certain steps in reasoning unnecessary. Hutchins (1993) documents the challenge for experienced crewmen on a Naval vessel to determine their location in San Diego bay when their instrumentation breaks down and they have to resort to earlier methods that require their coordinated engagement in problem solving. The example testifies to the adaptive abilities of the crew; but it also highlights how technology can shield us from the intricacies of problem solving. The spring-driven, and hence seaworthy, clock allowed navigators to tackle the age-old problem of determining their longitude at sea. There can be no doubt that it represented a significant achievement and there is some justification for Sobel’s (1995) reference to the wide availability of inexpensive timepieces as ‘The Mass Production of Genius’ (*op. cit.*, 152). But the wealth of knowledge and insight that went into the invention of highly reliable timepieces does not pass directly from the instrument to its users any more than citizens of the Western world since 1582 learn about astronomy by using a Gregorian calendar<sup>6</sup>.

An analogous claim is often made about the introduction of calculators into the mathematics curriculum. Opponents note that graphic calculators take over many of the functions—dividing, finding square roots, plotting, etc.—that students have traditionally been required to grapple with. In a sense, a calculator may remove students from the circumstances and mathematical issues that gave rise to this invention. Others may argue that by freeing students from the intricacies of computation they can focus on new mathematical issues that require judgment and critical thinking.

A study of the evolution of symbol systems helps to place such debates into the larger context of how representational systems shift attention from old issues to new. What mathematical properties come to the fore when a particular representation system—a number line, set theory, and algebraic notation—is called upon to represent and model ideas? What issues are peculiar to the representational system and how do universals or general ideas manifest themselves in these systems?

The affordances and constraints of representational systems are not fully contained within them but rather depend on how they are conceived and deployed.

This is why we need to look carefully at the nature of the activities students are engaging in and to recognize what skills and understanding have been implicitly delegated to the technology and what skills and functions students are effectively exercising.

As Goody (1977) noted in reviewing the effects of literacy: ‘writing is not a monolithic entity, an undifferentiated skill; its potentialities depend upon the kind of system that obtains in any particular society’ (p. 3). He then provides several examples of ‘restricted literacy’ where societies have failed to realize the full potentialities mentioned in his earlier work (Goody & Watt, 1963). Indeed, he confesses that the original work should have been entitled ‘The Implications of Literacy’ to avoid giving the undue impression that the technology of writing brings with it a set of predetermined effects.

If, as Kaput and Shaffer suggest, computational technologies do indeed increasingly become integral to how we represent and solve mathematical problems, we hope that, as some representational tasks are offloaded for the software to handle, educators will find new domains of inquiry for students to exercise their reasoning and creative energies.

#### NOTES

- <sup>1</sup> They might also include cases where one has made a generalization in thought but not yet outwardly expressed it, provided that the thought involves a sentence that, *if spoken*, would meet the first criterion.
- <sup>2</sup> Our motivation was similar to that of Kant in formulating the concept of scheme.
- <sup>3</sup> The fact that Wittgenstein and Cassirer were dealing with categorical relations, where class inclusion plays a heavy role, should not distract us from noting that they are searching for alternatives to approaches to concepts based on physical properties.
- <sup>4</sup> Other video segments show that some adjustment was necessary: she initially treated the train’s movements and her own body movements as having diverse effects on the shape of the graph.
- <sup>5</sup> We are using the *Pascal’s triangles* in a loose sense. In Pascal triangles, binomial coefficients occupy the positions of dots.
- <sup>6</sup> One could make a case for precisely the opposite effect: calendars obviate the need for attention to heavenly bodies to determine time of year.

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