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Integrating algebra and proof in high school mathematics: An exploratory study

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ABSTRACT

Frequently, in the US students' work with proofs is largely concentrated to the domain of high school geometry, thus providing students with a distorted image of what proof entails, which is at odds with the central role that proof plays in mathematics. Despite the centrality of proof in mathematics, there is a lack of studies addressing how to integrate proof into other mathematical domains. In this paper, we discuss a teaching experiment designed to integrate algebra and proof in the high school curriculum. *Algebraic proof* was envisioned as the vehicle that would provide high school students the opportunity to learn not only about proof in a context other than geometry, but also about aspects of algebra. Results from the experiment indicate that students meaningfully learned about aspects of both algebra and proof in that they produced algebraic proofs involving multiple variables, based on conjectures they themselves generated.

1. Introduction

Over at least the past decade, the learning and teaching of algebra has increasingly become a central component of the mathematics education research agenda (Gutiérrez & Boero, 2006; RAND, 2003; Stacey, Chick, & Kendal, 2004). Indeed, algebra is often considered as a gatekeeper to accessing, and ultimately understanding, more advanced mathematics (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2009; National Mathematics Advisory Panel, 2008). Despite the importance of algebra in school mathematics, researchers have demonstrated the difficulties students have encountered in learning algebra. For example, students often believe the equal sign only represents a unidirectional operator (e.g., Kieran, 1981, 1985; Vergnaud, 1988a) and do not recognize the commutative and distributive properties (e.g., Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 2001; MacGregor, 1996), while other studies show that students do not comprehend the use of letters as generalized numbers or as variables (e.g., Booth, 1984; Kuchemann, 1981) and have difficulty operating on unknowns (e.g., Bednarz, 2001; Filloy & Rojano, 1989; Kieran, 1989). Although the extant research highlights a range of areas in which students have difficulty learning algebra, the field currently lacks a sound body of research focused on challenges students face when learning more advanced algebraic concepts and skills.

Among other concepts and skills that were recommended by the National Mathematics Panel (2008), multiple variables and parameters come into play when more advanced algebra concepts are involved. Few studies have investigated the obstacles students face when working with multiple variables and parameters (Bloedy-Vinner, 1994, 1996; Furinghetti & Paola, 1994; Ursini & Trigueros, 2004), while even fewer studies address suitable conditions conducive to the learning

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of algebra when it involves multiple variables (Bardini, Radford, & Sabena, 2005; Binns, 1994; Sedivy, 1976). Variables and parameters are among the main uses of symbols in algebra. Previous research has highlighted that meaningful use of symbols is a central aspect of the learning of algebra (Kaput, Blanton, & Moreno, 2008). It involves carefully defining the meaning of symbols introduced to solve problems, including specifying units and distinguishing among the three main uses of symbols in algebra: (1) as unknowns (e.g., find the value of Q such that $3Q - 4 = 11$); (2) as variables, namely placeholders that can take on a range of values (e.g., $a + c = c + a$ for all a and c); and (3) as parameters (e.g., What is the effect of increasing m on the graph of $y = mx + b$?) (Usiskin, 1988). In summary, further research is needed to highlight the obstacles students face when working with multiple variables and, most importantly, to document students' learning process as they overcome those obstacles. Moreover, further research is needed to investigate the learning environments, problem situations, and conditions that can promote students' learning of more advanced algebraic concepts such as multiple variables[1] and parameters.

We similarly lack a robust research base focused on students' understanding of proof in high school in domains other than geometry. Indeed, Hanna (2007) argues that mathematical proof has increasingly played a less prominent role in the secondary mathematics curriculum in the United States (US), thus urging, "We need to find ways, through research and classroom experience, to help students master the skills and the understanding they need" (p. 15). Indeed, the void in the US curriculum regarding proof is surprising, with the exception of its inclusion in the high school geometry curriculum, mainly through its emphasis on the two-column format. Despite this past lack of emphasis on proof, however, there currently seems to be a shift in how proof is viewed as part of the secondary mathematics curriculum. In the newly released *Reasoning and Sense Making* (NCTM, 2009) document, proof is seen as an aspect of reasoning which is intertwined throughout both formal and informal observations and deductions: "reasoning and sense making are intertwined across the continuum from informal observations to formal deductions, despite the common perception that identifies sense making with the informal end of the continuum and reasoning, especially proof, with the more formal end" (p. 4). Moreover, the authors argue that proof should be naturally incorporated into all areas of the curriculum. Indeed, there are currently several documents advocating for a central role of proof in the teaching and learning of mathematics across all grades (e.g., Stylianou, Blanton, & Knuth, 2008). Further, some research suggests that integrating proof into domains other than geometry holds much promise for students' understanding of proof (e.g., Hanna & Barbeau, 2008; Healy & Hoyles, 2000; Pedemonte, 2008; Stylianou et al., 2008). Some caveats do exist, though, such as Herbst's argument (2002) that a change in how students view proof would require more than minor adjustments or calls for reform. Since we are far from having a sound body of research about proof in domains other than geometry, we still need to systematically study how to incorporate proof into areas other than geometry.

In this paper, we hypothesize that *algebraic proof* can serve as a vehicle for integrating more advanced algebraic concepts (e.g., problems involving multiple variables) together with mathematical proof, thus supporting students' understanding of both of these important mathematical topics. In fact, some researchers have highlighted that proofs are more than instruments to establish that a mathematical statement is true. Indeed, they embody mathematical knowledge in the form of methods, tools, strategies, and concepts (Hanna & Barbeau, 2008; Rav, 1999). Rav (1999) argues that this is one of the reasons why mathematicians re-prove theorems. Viewing theorems as tools through which mathematical knowledge is created (Hanna & Barbeau, 2008) is central to integrating proof into other areas of the secondary curriculum. Thus, the field is in need of studies that develop ways of integrating proof in this broader way, across different areas of mathematics and across grade levels, and to systematically study the ways in which students engage with innovative problems that involve algebraic proof.

It is in this context that we present findings from a teaching experiment that focused on an integrated approach to the teaching of algebra and proof in high school. Accordingly, the goals of this paper are as follows:

1. To introduce our working hypothesis that an integrated approach to algebra and proof has great potential to foster students' meaningful learning of both algebra and proof.
2. To make explicit the relationship between the problem used in the teaching experiment and the design principles used to inform the experiment.
3. To describe students' learning process while they were engaged in a teaching experiment that integrated algebra and proof. Specifically, we identify and characterize each stage in students' problem solving process through a detailed qualitative analysis of students' prevailing activities (e.g., selecting variables, conjecturing, applying the distributive property, etc.). In parallel, we pay close attention to whether mistakes, obstacles, and misconceptions that have been reported in previous studies (e.g., the equals sign only represents a unidirectional operator that produces an output on the right side from the input on the left) emerged as part of students' problem solving process.

2. Framework

2.1. Algebra: a tool with the potential to make explicit what is implicit

Past research has demonstrated the many difficulties students often face when learning algebra. For example, research has shown that students do not seem to understand that the equal sign indicates a relationship between the quantities on

[1] In this context, we use the term multiple variables to describe mathematical problems that involve more than two variables. For instance, problems involving functions on one variable (with an independent variable and one dependent variable) and systems of linear equations (with two unknowns).

both sides of the sign, but rather believe that it only represents a unidirectional operator that produces an output on the right side resulting from the input on the left (e.g., Booth, 1984; Kieran, 1981, 1985; Vergnaud, 1985, 1988a, 1988b). Other research has demonstrated that students often focus on finding particular answers (e.g., Booth, 1984), do not recognize the use of commutative and distributive properties in their work on algebra (e.g., Boulton-Lewis et al., 2001; Demana & Leitzel, 1988; MacGregor, 1996), and do not use mathematical symbols to express relationships among quantities (e.g., Vergnaud, 1985; Wagner, 1981). Still other researchers have found that students often do not comprehend the use of letters as generalized numbers or as variables (e.g., Booth, 1984; Kuchemann, 1981; Vergnaud, 1985), have difficulty operating on unknowns (e.g., Bednarz, 2001; Bednardz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg, Sleeman, & Ktorza, 1990), and often fail to understand that equivalent transformations on both sides of an equation do not alter its truth value (e.g., Bednarz, 2001; Bednardz & Janvier, 1996; Filloy & Rojano, 1989; Kieran, 1985, 1989; Steinberg et al., 1990). Together, this body of research illustrates that students have difficulty engaging with algebraic concepts typical of Algebra I courses. In general, these studies report on students' learning when working on problems that involve two variables at most.

In response to some of these difficulties mathematics education researchers have proposed different ways of looking at school algebra in an effort to foster students' meaningful learning. For instance, algebra centered on the concept of function and the entailed notion of variable, known as the functional approach (e.g., Schwartz & Yerushalmy, 1992), has put forward the idea that instead of starting with unknowns and equations, school algebra should start with the notions of variable and function, and later introduce the notion of unknown and equation as the specific case where two functions intersect, i.e., x such that $f(x) = g(x)$. A focus on algebra as generalization (Lee, 1996; Mason, 1996) constitutes another example, where algebraic notation is used to express generalizations of arithmetic or geometric patterns.

In this paper, we propose another approach that also aims to offer another way of looking at school algebra, with the same goal of fostering students' meaningful learning of algebra. From our perspective, what is still missing in previous research is an emphasis on one of the most important features of algebra: by manipulating an expression we can *read* information that was not *visible* or *explicit* in the initial expression. For instance, using algebra we can show that if we add three consecutive integer numbers, $a + (a + 1) + (a + 2)$, the sum will always be a multiple of three, $3a + 3$. We can also show how we moved from the initial expression to the final expression using the properties of distribution and associativity: $a + (a + 1) + (a + 2) = (a + a + a) + (1 + 2) = 3a + 3$. In addition, we can also see that the sum will always be the triple of the second number $3(a + 1)$ *by factoring* 3 *from both terms*. This aspect of algebra — the use of algebraic notation to make explicit what was previously implicit — has great potential to link algebra with proof due to its capacity to unveil that a certain property holds for all cases (e.g., the sum of any three consecutive integers is [*always*] a multiple of three). The approach adopted in the study here reported is original in that this feature of algebra is central to our teaching experiment.

2.2. Algebra as a modeling tool

Given our focus on integrating algebra and proof, Chevallard's (1989) framework is of special interest because of his conceptualization of algebra as a modeling tool. In his framework, algebra is envisioned as a tool to mathematically model problems. Algebra helps us to study a situation by defining and denoting variables and parameters, and the relations among them. In addition, and most importantly, once these have been defined, algebra provides us with the tools to transform an algebraic expression into other equivalent expressions (e.g., using associativity) with the potential to reveal something that was *hidden* or *implicit* in the original expression. Chevallard (1989) described the modeling process using algebra in the following way:

> (1) We define the system that we want to study by identifying the pertinent aspects in relation to the study of the system that we want to carry out, in other words, the set of variables through which we decide to cut off from reality the domain to be studied . . . (2) Now we build a model by establishing a certain number of relations R, R', R'', etc., among the variables chosen in the first stage, the model of the systems to study is the set of these relations. (3) We 'work' the model obtained through stages 1–2, with the goal of producing knowledge of the studied system, knowledge that manifests itself by new relations among the variables of the system. (p. 53)

Recalling the example presented in the previous section, we can use algebra to show (i.e., prove) that the sum of any three consecutive integer numbers is *always* a multiple of three. First, following Chevallard, we identified the variables which are the three integers a, b, and c. Since they are consecutive we can, as outlined in the second step, identify the mathematical relations that are at play. Thus, $b = a + 1$, and $c = b + 1$. In addition, since, $b = a + 1$ substituting in $c = b + 1$ gives us $c = (a + 1) + 1 = a + 2$. So far, we have expressed the three consecutive numbers by using the "consecutive" relation obtaining the following: a, $a + 1$, $a + 2$. Now, in the third stage, since we have the same variable, we can *work* the model as follows: $a + (a + 1) + (a + 2) = (a + a + a) + (1 + 2) = 3a + 3 = 3(a + 1)$. This last expression shows that, in fact, the sum of three consecutive integers is (always) a multiple of three. In addition, we can also say with precision that the sum is a specific multiple of three; it is also a multiple of the second consecutive number. As discussed previously, this aspect of algebra, namely the potential to *reveal* or make *explicit* new information through the use of properties, has been to date underplayed in school algebra.

Furthermore, we believe that this aspect of algebra has a great potential for connecting algebra and proof. In fact, using algebra we can generalize patterns and represent relations. Therefore, we can capture all cases with a general expression. This is necessary when we need to prove a statement with a universal quantifier. In addition, using algebra we can manipulate and transform an expression into equivalent expressions (e.g., using distributive property); as a consequence, we may make explicit in the later expression what we wanted to show (i.e., prove).

In sum, two aspects of algebra that are central in our approach have a great potential to connect algebra and proof. First, algebra as a modeling tool allows us to represent quantities and relations. Second, by transforming expressions we can make explicit information that otherwise was implicit in the initial form of the expression. These two aspects of algebra are at the forefront when constructing *algebraic proofs*.

2.3. Mathematical proof

Regarding proof, our framework has been conceptualized bringing together a range of prior research. From the work of Balacheff (1982, 1988), we build on the notion that *proof* is an explanation that is accepted by a community at a given time. This notion of proof is understood together with that of *explanation* and of *mathematical proof*. An *explanation* is the discourse of an individual who aims to establish for somebody else the validity of a statement. The validity of the statement is initially related to the speaker who articulates it. A *mathematical proof* is a proof accepted by mathematicians. As a discourse, mathematical proofs have now-a-days a specific structure and follow well defined rules that have been formalized by logicians. In the teaching experiment described in this paper, we centered our work on the notion of *proof* in Balacheff's sense. For example, whenever students used a finite set of examples to show that a universally quantified (i.e., for all) statement was true, we considered it as an *explanation* but not as a proof. Therefore, in our teaching experiment we focused on students shifting from relying *solely* on empirical evidence (i.e., producing explanations) towards relying on mathematical properties and using deductive reasoning (i.e., producing proofs).

We also drew from Hanna's (1990) distinction between *proofs that (just) prove* and *proofs that (also) explain*. The first type *only* establishes the validity of a mathematical statement while the second type, in addition to proving, reveals and makes use of the mathematical ideas that motivate it. In a similar vein, Arsac et al. (1992) proposed three roles for proofs as part of an instructional task: to understand why and/or to know, to decide the truth-value of a conjecture, and to convince oneself or someone else. Consequently, we adopted this broader view of the role of proof that goes beyond establishing the truth-value of a statement. Indeed, we adopted as a design principle that a proof has the potential to help students understand why a specific phenomenon happens. Therefore, in the problems that were implemented in the teaching experiment, the role of proof was not only to establish the truth-value of a statement but also to foster students' understanding of why a specific mathematical phenomenon happens.

Another critical aspect of our work, following Boero, Garuti, and Lemut (2007), is that conjecturing and proving are interrelated and crucial mechanisms in generating mathematical knowledge. Therefore, in the problems analyzed in this paper, students were not provided with the conjecture to prove. Instead, as part of the problem they had to construct or produce their own conjectures, and then prove them.

In summary, in our teaching experiment, proof was conceived as an explanation accepted in the classroom community at a given time (Balacheff, 1982, 1988). The focus was on proof as an explanation that shows why a particular mathematical phenomenon happens (Arsac et al., 1992; Hanna, 1990). In addition, and building on Boero's work (2007), the problems were not introduced as "proving tasks" or "proof problems." Students were given an open-ended problem, and as a result of their exploration, they produced conjectures. Only after that would the teacher (i.e., the first author of this paper) prompt them to "show why this is always true."

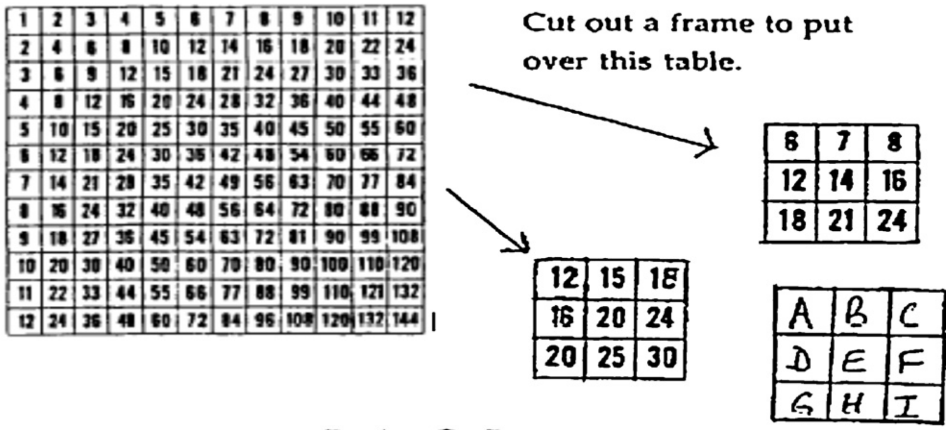
2.4. Tension between empirical and theoretical evidence when proving a mathematical conjecture

Students' strong inclination to prove using empirical evidence (i.e., using a finite set of examples) to prove a universal (i.e., for all) mathematical conjecture has been reported recurrently in previous studies. For instance, Harel and colleagues (Harel, 2007; Harel & Sowder, 1998; Martin & Harel, 1989) conducted several studies with the aim of developing and refining students' conceptions of proof, namely *proof schemes*. These studies included undergraduate students, mathematics majors, and preservice elementary teachers. Repeatedly, their results ratified the existence of two classes of proof schemes: the *empirical* and the *deductive*. The former is characterized by its reliance on either evidence from examples or perceptions. The latter is characterized by the generality of the proof, and students' use of operational thought and logical inference.

In a similar vein, Balacheff (1988) proposed two categories to classify middle school French students' proofs: *pragmatic* and *conceptual*. "Pragmatic proofs are those having recourse to actual actions or showings; in contrast, conceptual proofs are those which do not involve action and are based on formulations of properties in question and relations between them" (p. 217).

Although there are differences between both categorizations (empirical and deductive for Harel; pragmatic and conceptual for Balacheff), both distinguish students' proofs on the basis of whether or not students rely on empirical evidence rather than on mathematical relations and logic. Also, both categorizations found a strong inclination among students to use examples to prove. Therefore, in our teaching experiment, we made use of students' strong inclination to use numeric examples, and put it at the service of exploring the problem and producing conjectures. At the same time, students were challenged to provide evidence grounded on mathematical relations and logic. Thus, our intention was that students would use examples to explore the problem and, as a result, they would come up with conjectures about different aspects of the problem. At this point, it was part of the work of the teacher to challenge them to produce evidence that would show that their conjecture was true for all cases, not only for the particular examples they had explored, and to explain why.

CORNERS AND MIDDLES



B - A = C - B

(the B number - the A number) = (the C number - the B number)

Check that this is true in both the boxes shown.
Check other places on the table.

So B - A = C - B is *always* true

- Consider A + D = F + I
  This is *never* true. Check this.
- Consider E = 2A
  This is *sometimes* true. Check this.
- With your partner, make a list of 6 relations and challenge another pair to decide whether they are always, never or sometimes true. But sure of your answers first.
  Try to get some of each type.

Fig. 1. Corners and Middles (Bell, 1995, p. 13).

In sum, we need to study the design and implementation of proof-eliciting problems[2] (Harel & Lesh, 2003) across mathematical domains, other than geometry, that acknowledge students' inclination to use examples. One way of doing this is to offer students the opportunity to engage in problems that require the construction of a proof through an approach in which students are not required to prove up front, but rather they have to first investigate a mathematical phenomenon and, as a result, produce conjectures. Students would engage in a "proving situation" framed as coming up with evidence beyond particular cases that would show that the conjecture is *always* true, and also give us insight as to *why* this is the case. It is precisely these types of problems that are at the center of the teaching experiment discussed in this paper.

2.5. Prior studies integrating algebra and proof

Some researchers have developed mathematical tasks integrating algebra and proof, and studied their effect on students' learning. For instance, Bell (1995) reported that mathematical tasks such as "Corners and Middles" (see Fig. 1) have proven to be an effective way of getting students "to use algebraic language in situations where it forms a natural means of communication. Note that opportunities for checking, and understanding the possibility of relations being true always, sometimes, or never, are built in also" (p. 12).

[2] Harel and Lesh (2003) coined the term proof-eliciting problems to describe problems through whose solution students gradually modify their ways of thinking regarding what constitutes proof.

Table 1
Sequence of problems intended to provide an opportunity to parameterize days in a week.

Lesson number	Problem number	Generic square including independent and dependent variables	Dimension of the calendar square	Length of the week (parameter)
1 and 2	1	$\begin{matrix} a & a+1 \\ a+7 & a+8 \end{matrix}$	2×2	7 is a specific instance of the parameter
13	9	$\begin{matrix} a & a+1 \\ a+9 & a+10 \end{matrix}$	2×2	9 is a specific instance of the parameter
13	18	$\begin{matrix} a & a+1 \\ a+d & a+d+1 \end{matrix}$	2×2	d
Interview #2	12	$\begin{matrix} a & a+1 & a+2 & a+3 \\ a+d & a+1+d & a+2+d & a+3+d \\ a+2d & a+1+2d & a+2+2d & a+3+2d \\ a+3d & a+1+3d & a+2+3d & a+3+3d \end{matrix}$	4×4	d

In a similar vein, Friendlander and Hershkowitz (1997) referred to this type of activity as *generalization–justification* tasks given that they elicit mathematical generalizations and justifications. They report that these problems increase students' motivation and their understanding of the use of algebraic notation and mathematical proof. Barallobres (2004) reported similar favorable findings with a group of eighth graders working on a problem that elicited students' conjectures, and then asked students to explain why using algebra.

Based on these findings, we hypothesize that an integrated approach to algebra and proof has the potential to foster students' meaningful learning of both algebra and proof.

3. Methodology

3.1. Design principles

In this section, drawing from the aforementioned studies, we synthesize the principles that guided the design of the Calendar Algebra Problems, which were the focus of the teaching experiment:

(1) to promote students' production of conjectures (Boero et al., 2007) and the entailed interplay between examples and counterexamples;
(2) to focus on algebra as a modeling tool (Chevallard, 1989) by:
  a. defining variables and parameters,
  b. defining relations among the previously defined variables and parameters,
  c. applying properties and producing equivalent expressions;
(3) to highlight the role of proof as centered around understanding why something happens and not just to establish truth-value, following Arsac et al. (1992) and Hanna (1990);
(4) to highlight that the link between algebra and proof lies in students' production of equivalent algebraic expressions to *reveal* or make *explicit* information in the expression to understand why something happens.

In the following section, we discuss how these design principles were used to inform the design of the Calendar Algebra Problems used in the teaching experiment.

3.2. Calendar Algebra Problems

Problems in this teaching experiment share the same calendar context, and were presented according to an increasing degree of complexity. For example, in Problem 1, students worked on a regular calendar with seven days per week and a 2×2-calendar square. In this case, the multiple variables (one independent variable and the rest dependent variables) correspond to the numbers (i.e., day-number) within the 2×2-calendar square. In problem 9, the length of the week changed from 7 to 9 days (see Table 1). Consequently, the level of difficulty increased given that students had to define not only the multiple variables involved in the problem but also the mathematical relations among the variables changed (i.e., a, $a+1$, $a+9$, and $a+10$). This change in the length of the week was intended to lay the groundwork for problem 18, in which the length of the week was parameterized to d days (see Table 1[3]). As mentioned before, in this article, we will report on students'

[3] In Table 1, Problems 1, 9, 18 and 12 illustrate the type of problems that students worked on during this teaching experiment. We were not able to include all problems given space restrictions.

Problem 1

Part 1. Consider squares of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be $\begin{smallmatrix} 1 & 2 \\ 8 & 9 \end{smallmatrix}$. These squares will be called 2x2-calendar-squares.

Calculate:

1. The product between the number in the upper left corner and the number in the lower right corner.
2. The product between the number in the upper right corner and the number in the lower left corner.
3. To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

Find the 2x2-calendar-square that gives the biggest outcome. You may use any month of any year that you want.

Part 2. Show and explain why the outcome is always going to be -7.

Fig. 2. Problem 1 from the Calendar Algebra Problems.

mathematical work on Problem 1, which is one of the seventeen problems that are part of the Algebra Calendar problems. The ultimate goal of the sequence of problems was to include two parameters in addition to the multiple variables; that is, to parameterize both the length of the week (i.e., d) and the dimension of the calendar-square (i.e., $n \times n$).

Fig. 2 shows Problem 1 from the Calendar Algebra Problems; students worked on this problem during the first two lessons of the teaching experiment.

In the first part of Problem 1, students need to determine where to place the 2×2-calendar-square to obtain the largest outcome. Note that the problem was purposely framed as finding the "largest outcome," as opposed to "prove that the outcome is always -7." In this way, the problem remains open (Boero et al., 2007), allowing students to explore and discover by themselves any potential regularities. Students need to figure out what is going on in the problem; it is not given to them. It was expected that students would anticipate that there is a place where the outcome is the largest, when actually there is not. After trying several examples their findings would indicate that the outcome is -7, contradicting their expectations. From our point of view, it was expected that as a result of their exploration of the first part of the problem, students would produce conjectures about the behavior of the outcome. At this point, the teacher would write the conjectures produced by each of the groups, and re-launch the work as gathering evidence that will show that the conjecture is *always true*, and also to explain *why*.

Therefore, in the second part of Problem 1, the task for the students was to figure out why this happens (i.e., why the outcome is -7), and whether this is "always" going to be the case. It is at this stage that students engage in the proving phase. Based on previous studies (e.g., Arsac et al., 1992), students usually over-generalize a result that they observe to be true in a finite non-exhaustive set of cases. Thus, one of the challenges in the second part of Problem 1 is to show the limitations of this strategy (i.e., using a finite non-exhaustive set of examples to prove that a universal proposition is true) and to encourage students to use algebra as a *tool* that allows them to express an infinite set of numbers using an expression.

Since we consider that variables, algebraic expressions, and equivalent expressions are at the core of school algebra (e.g., Boero, Garuti, & Mariotti, 1996; Mason, 1996; Usiskin, 1988), a goal of this teaching experiment was to promote the following activities: choosing relevant variables, setting up relations among variables and/or parameters, expressing a set of variables as a function of one of the variables, making decisions about transformations (i.e., which transformations to apply), transforming into equivalent expressions, working with equivalent expressions, deciding when to stop transforming, and reading and interpreting the chosen final expression in terms of their hypothesis and the context.

In Problem 1, a model (Chevallard, 1989) of the 2×2-calendar-square could look like the one shown in Fig. 3.

Using the model shown in Fig. 3 as a starting point, the outcome and a possible chain of equivalent expressions are shown in Fig. 4. Our intention was that in producing a chain of equivalent expressions, students would use one of the aspects of algebra that was at the center of this teaching experiment as articulated in design principle: to make explicit something that was implicit in the initial algebraic expression. In fact, nothing indicates in the initial expression (i.e., $a(a+8)-(a+7)(a+1)$) that the outcome is -7. Moreover, only by producing a chain of equivalent expressions can we arrive at that conclusion.

3.3. Participants

In this teaching experiment, a group of 9 high school students (9th and 10th graders) participated in fifteen one-hour-long lessons that took place in a charter public high school in the Boston area, Massachusetts, US. The group of students was diverse in terms of their mathematical performance. There were four female (Abbie, Desiree, Grace, Audrey) and five male

$$\begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix} \equiv \begin{matrix} a & a+1 \\ a+7 & a+8 \end{matrix}$$

Fig. 3. Model of a 2×2-calendar-square.

$$
\begin{aligned}
a(a+8)-(a+7)(a+1) &= a^2+8a-(a^2+a+7a+7)= \\
&= a^2+8a-a^2-a-7a-7= \\
&= a^2-a^2+8a-a-7a-7= \\
&= -7
\end{aligned}
$$

Fig. 4. Transformation of the initial outcome into equivalent expressions.

(Chris, Janusz, Brian, Cory, Tyler) students. They worked in the same groups of three throughout the teaching experiment. Abbie, Desiree, and Grace were in one group; Brian, Cory, and Tyler in another group; and Audrey, Chris, and Janusz were in a third group.

3.4. Data collection

From the fifteen lessons that comprised the teaching experiment, we will only report on data collected during Lessons 1 and 2 (i.e., Problem 1) that focused on algebraic proof involving multiple variables. Problems used in the second part of the teaching experiment involved not only multiple variables, but also multiple parameters.

3.5. Lessons

Fifteen one-hour lessons were implemented once a week. These lessons were part of the regular school schedule but not part of the students' regular mathematics classes. Each lesson was video and audiotaped, and the written work was collected and scanned for analysis. The first author of this paper was the teacher researcher throughout the teaching experiment.

3.6. Individual interviews

Students were interviewed individually twice during the intervention, once halfway through the experiment and once at the conclusion of the teaching experiment. In these interview sessions, students were asked to solve a problem that was new to them, but similar to the problems discussed during the lessons. The purpose of the interviews was to develop a deeper understanding of students' problem solving processes.

During the first interview, students were asked to solve a problem similar to Problem 1 in which instead of having a 2×2-calendar-square they had to study the situation for a 3×3-calendar-square. Also, the calculation involved different elements, described as follows:

> Consider two elements: (1) the product between the number in the center of the first row and the number in the center of the third row; and (2) the product between the number in the center of the first column and the number in the center of the third column. To the number obtained in (1) subtract the number obtained in (2). This result is your outcome.

The other features of the problem were kept constant: the length of the week was 7 days, and the outcome was constant, namely -48. Eight students out of nine were able to prove their conjecture using algebra; the one student who did not finish the proof was not able to simplify one of the expressions. The analysis of the data is currently underway and will be published in an upcoming paper. The second interview involved a more complex problem since it required parameterizing the length of the week; thus, in addition to the multiple variables, students needed to incorporate a parameter in the mathematical model representing the situation. Analysis of data corresponding to this interview is currently underway.

4. Data analysis

Data was analyzed qualitatively taking a grounded theory, bottom-up approach (Glaser & Strauss, 1967). In other words, starting from the data, theoretical relationships and categories are constructed. Our analysis suggests that students engaged in varied mathematical activities in the process of solving Problem 1. These mathematical activities were focused on accomplishing a specific goal: conjecturing or proving. Our analysis suggests that the challenges that students faced were intrinsic to the goal (i.e., conjecturing or proving) that they were trying to accomplish. Thus, we use each of the goals orienting students' activities to define the different stages through which students progressed as they solved Problem 1: (1) Conjecturing and (2) Proving. Moreover, we identified distinct sub-stages at the interior of the Proving Stage. Each of these sub-stages corresponds to a sub-goal within the larger goal of *proving*: (2a) Shifting from numeric examples to algebra; (2b) Constructing a generic square; (2c) Setting up an initial expression; (2d) Generating a chain of equivalent expressions; and (2e) Interpretation of the last expression. Thus, the results section is structured based on these stages and sub-stages.

5. Results

Overall, students were engaged while solving Problem 1 in that the problem seemed to be accessible for students while also presenting new challenges for them. The analysis indicates that the problem worked as intended during the design phase. In fact, students were able to use algebra as a tool to model the mathematical relations in the problem. Moreover, students were able to use algebra to prove their own conjectures.

According to our analysis, students faced the following challenges as they worked on Problem 1:

- Shifting from using numeric examples towards using algebra to prove.
- Defining the independent and dependent variables.
- Shifting from using algebra in an equation solving mode towards using algebra to deductively generate equivalent expressions to ultimately prove their conjecture.

Mathematical aspects related to the above challenges were discussed among students and negotiated with the teacher; as a result, students were able to overcome them. These discussions took place both in small and large group contexts. In what follows, we describe in detail what occurred in terms of students' mathematical processes at the interior of each of the stages identified above.

5.1. Stage 1: conjecturing

As discussed earlier, one of the goals of the teaching experiment was to promote students' construction of their own conjectures, as opposed to the more traditional approach where students are given an already-made conjecture to prove, typical of high school geometry courses in the US. As intended, Problem 1 provided students with the opportunity to produce their own conjectures about the behavior of the outcome. In fact, initially students produced a variety of conjectures: (a) the square should be placed at the end of the month (where the biggest numbers are) in order to obtain the biggest outcome; (b) the square should be placed at the beginning of the month (where the smallest numbers are) in order to obtain the biggest outcome because there is a subtraction within the calculation; and (c) it does not matter where the square is located since the outcome will be always -7. Finally, after trying with more examples, all students agreed on the same conjecture, stating that the outcome is -7 no matter where the 2×2-calendar-square is placed. An in depth analysis of the conjecturing process has been reported elsewhere (Martinez & Li, 2010).

5.2. Stage 2: proving

5.2.1. Stage 2a: shifting from numeric examples to using algebra

Without evaluating whether the conjectures were correct or incorrect, the teacher prompted students to gather evidence that would help prove that their conjectures were in fact true. As anticipated, a majority of the students resorted to using numeric examples. The teacher used two strategies to move students away from using examples as the sole tool to prove the conjecture. The first strategy consisted of asking students whether the examples they had tried showed that the outcome corresponding to year 3067, for instance, would be -7. The second strategy intended to link the proof with understanding why this phenomenon (outcome equals -7) happens.

The exchange below illustrates the first strategy in a discussion among the members of one of the groups (Brian, Cory, and Tyler) and the teacher.

Teacher: . . . if somebody comes to you and they show you a calendar of the year 1920, let's say October 1920, how can you be sure that this is going to work?

. . .

Teacher: OK, let's try something then. Can I write in this one? [Here she drew a 2×2-square without the contents]
Tyler: Yes.
Teacher: How can you show that for any number this is going to be the case?
Tyler: Oh, so, you do like *x*.
Teacher: Ok, you can try.
Tyler: Right, so use like different variables. . . *a*, *b*, *c*, *d*.
Teacher: . . . why did you come up with the idea of using letters?
Tyler: Using variables?
Teacher: Yes, variables.
Tyler: Because if you have a formula then it shows that it will always be the same. . .then.
Teacher: Aha.
Brian: Yeah.

Fig. 5. Generic 2 × 2-calendar-square. Desiree (left) and Chris (right).

Tyler: Mmm... then ... then...it will work for anything, right so actually.
Teacher: OK, try to work with that and see if that gives us something solid to prove.

In the group including Abbie, Desiree, and Grace, the idea of using letters was promoted by Abbie, as indicated in the exchange below. The exchange happened after they made a decision of first working individually and then sharing their ideas. Abbie takes the lead:

Teacher: ... wait a minute... why did you start using x? How come? You worked with numbers and suddenly I come and you are using x. I'm not saying it is wrong, I just want to know.
Desiree: Why did we jump to algebra?
Teacher: Yes.
Desiree: Because it's a variable; so you can plug in any number in here so instead of using 1, 2, 8, 9 [referring to $\begin{matrix}1 & 2\\ 8 & 9\end{matrix}$] we could use 6, 7, 13, 14 [referring to $\begin{matrix}6 & 7\\ 13 & 14\end{matrix}$] and it would still work for the problem as long as you plug in the right variables.

By the end of this stage, students were aware that they needed to use algebra in order to show that the outcome is always −7. However, they still had many challenges ahead to sort through as described in what follows.

5.2.2. *Stage 2b: the generic 2 × 2-calendar-square*

As part of their work, all students used what Desiree coined a *generic square* (Fig. 5), that is, a placeholder where students represented the multiple variables involved in the problem and the mathematical relations among them.

Only one of the three groups (Chris, Janusz, and Audrey) used from start to finish one independent variable and three dependent variables (Fig. 5, right). This group did not face any major challenges in constructing the generic 2 × 2-calendar-square.

The other two groups starting using more than one independent variable. The first group (Brian, Cory, and Tyler) initially started working with 4 independent variables (i.e., $\begin{matrix}a & b\\ c & d\end{matrix}$). Then, they came to realize the relationship between a and c, and b and d, that yields $\begin{matrix}a & b\\ a+7 & b+7\end{matrix}$. Later, these students saw the relationship between a and b, and came up with the last version of their generic 2 × 2-calendar-square $\begin{matrix}a & a+1\\ a+7 & a+8\end{matrix}$ (Fig. 6). This process is illustrated in the excerpt below:

Brian: It should be multiplied: a times d, and b times c [referring to $a \cdot d$ and $b \cdot c$ and to a model such as this one: $\begin{matrix}a & b\\ c & d\end{matrix}$]
Tyler: No, no, don't use c and d, use $a+7$ and $b+7$ [referring to a model such as the following: $\begin{matrix}a & b\\ a+7 & b+7\end{matrix}$]. Because...
Cory: Oh, yeah! It's a week away. Yes, he's right.

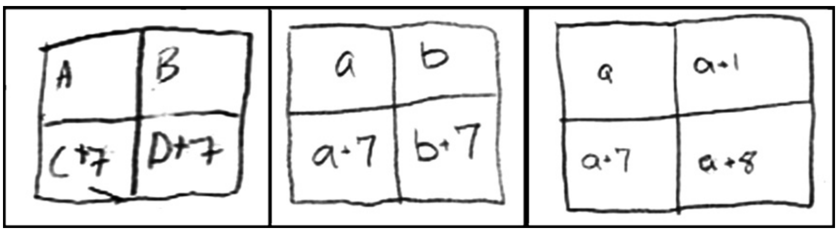


Fig. 6. Brian, Cory, and Tyler's process of production of the generic 2 × 2-calendar-square.

Tyler: Because it's the same thing plus 7. It is whatever this number is [the numbers in the upper cells of the model] plus 7. Since it is a week away, it will be +7.
Cory: Cause they are a week away.
Tyler: OK, so. . .we think we got this!

. . .

Tyler: This one is plus six cause is one day apart. . .
Brian: Yeah. That's one day ahead [the lower right cell], and that's one day behind [the lower left cell].
Tyler: Wait. Maybe we can . . . use a in all of them!
Cory: This [the lower right cell] is one day ahead. . .
Tyler: No, it's more than that.
Brian: It's 8 days ahead.
Cory: Yeah. . .of a week. And this [the lower left cell] is always one behind.
Tyler: Wow, so this. . . wait, wait, wait. . . I'm gonna do another one. . . Couldn't it be a, $a+1$, $a+7$, $a+8$? [referring to a model such as this one: $\begin{matrix} a & a+1 \\ a+7 & a+8 \end{matrix}$].

In sum, students in this group first used 4 independent variables (i.e., a, b, c, d). After recognizing the "+7" relationship between days in the same column, they reduced the number of independent variables to two with two dependent variables (i.e., a, b, $a+7$, $b+7$). Lastly, as a result of recognizing the "+1" relationship, they reduced the number of independent variables to one (i.e., a) and three dependent variables (i.e., $a+1$, $a+7$, $a+8$).

The remaining group that included Abbie, Desiree, and Grace started working with two independent variables (i.e., x and y) and two dependent variables (i.e., $x-8$ and $y-6$). Later they shifted to using one independent variable, becoming aware of the relationship between two consecutive days as illustrated in the excerpt below:

Abbie: All right, so this is what I have: 'x' times 'x minus 8' [referring to the product between the elements in one of the diagonals in this square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$], minus '$y$' times '$y$ minus 6' [referring to the product between the elements in the other diagonal in the same square $\begin{matrix} x-8 & y-6 \\ y & x \end{matrix}$]. That's basically what we're doing, yes. Let's check.

. . .

Desiree: Just plug in random numbers?
Abbie: Yeah.
Desiree: Except no, like over 30, but I bet that if you go over 30 it wouldn't. . .
Abbie [talking out loud while she is doing the calculation :]. . . times 12 minus 8. Oh, wait!
Grace: What?
Abbie: y and x are also related because x is one less than y. OK, so we can go x minus 1 times x minus 1 minus 6 [referring to $(x-1)\cdot(x-1-6)$ and to a square such as the following: $\begin{matrix} x-8 & x-1-6 \\ x-1 & x \end{matrix}$]. So, y and x are not like random numbers because if here it is 17 that would be. . . So, we can say everything in terms of this number right here [referring to the anchor variable in her model, the x in the lower right cell].

In sum, all students produced a generic 2×2-calendar-square when solving Problem 1 where the multiple variables and mathematical relations were represented. Two of the three groups faced the same challenge: to define the multiple variables involved in the problem, and to determine which of them were independent and which were dependent. We hypothesize that the context of the problem (i.e., the calendar) was helpful for these students to overcome this challenge.

5.2.3. Stage 2c: setting up the initial expression representing the outcome

Once students produced the generic 2×2-calendar-square, they used it to generate the *generic* outcome as described in the problem (i.e., subtraction of the cross product). This is illustrated in the excerpt below from the group that included Brian, Cory and Tyler:

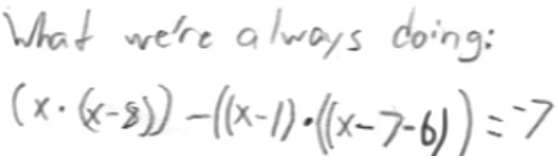


Fig. 7. Grace's Initial Expression (Equation Form).

Tyler: Then, it would be a, $a+8$, right? Minus... [referring to $a \cdot (a+8)-$]
Cory: $a+7$...
Tyler: No, $a+1$.
Cory: $a+7$ [completing the expression to become $a \cdot (a+8) - (a+1) \cdot (a+7)$].
Tyler: Equals...
Cory: -7 [further completing to $a \cdot (a+8) - (a+1) \cdot (a+7) = -7$]...
Tyler: Well, that's what we want the outcome to be, right?
Cory: Watch, try it.
Tyler: Well, that is what it should be. Equals -7. All right so, fill it in with numbers. [The students then tried with $a = 1$. They decided that their expressions were correct after trying the numbers.]

This excerpt illustrates that the 2×2-calendar was instrumental for students to write the initial expression; this was the case in all three groups of students participating in the teaching experiment. In addition, it can be observed how the teacher, by asking "Is there any way that we can work this out?," was helping students to advance their inquiry without telling them exactly what to do.

A finding that was unexpected and took the teacher by surprise was the use of "-7" after the equal sign in the initial expression (Fig. 7). All students except for one (i.e., Janusz, Fig. 8) included -7 after the equal sign in the initial expression.

To describe these two types of initial expressions, we coined the terms "Equation Form" (Fig. 7) and "Algebraic Expression Form" (Fig. 8). The former describes the initial expression that includes "-7" after the equal sign. The latter describes the initial expression that does not include "-7" after the equal sign. Looking back, perhaps it should not have been surprising that most students used an *equation*, instead of an *algebraic expression*, form, given that students were familiar with variables as part of an equation either in an equation solving context or functional context. Indeed, what was new for them was not to add anything after the equal sign.

In sum, all students were able to represent the initial expression using algebra; having the generic 2×2-calendar-square was central in students' success. Students represented the initial expression in two forms: "Equation" and "Algebraic Expression," the former includes "-7" after the equal sign while the latter does not.

5.2.4. Stage 2d: producing a chain of equivalent expressions

Once students produced an *initial expression*, they faced several challenges. The first challenge was to figure out what they could possibly do with that initial expression. The following discussion between Abbie and Desiree is an illustration of this challenge:

Abbie: I don't know how to prove things like this.
Desiree: I can help you with that but I have a question, the fact that this is $x-7$ [referring to a part of their expression which is $x \cdot (x-8) - (x-1) \cdot (x-7) = -7$] has something to do with the fact that the answer is -7?
Abbie: Ah...

[After a while Abbie states:]

Abbie: I think that we need to do it algebraically instead of thinking it through.

[Then she continues:]

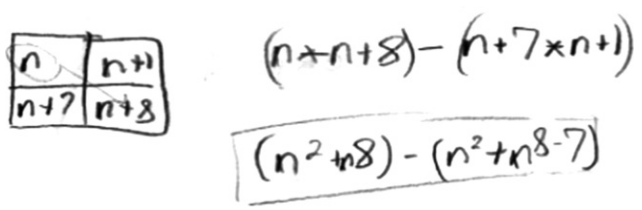


Fig. 8. Janusz's Initial Expression (Algebraic Expression Form).

Abbie: This is backwards for me because usually I'm doing something and working towards making a formula and now I have a formula and I have to prove that it is true.
Desiree: This is part of why we are here. . .

Facing the same challenge, in Brian, Cory, and Tyler's group the teacher intervened to push for the idea of working on the *initial expression*, as illustrated in the following exchange:

Teacher: Ok, you made a lot of progress. What you want to show here is whether this is always going to be −7.
Tyler: Right.
Teacher: Is there any way that we can work this out?

. . .

Tyler: . . .we can cancel out both *a*'s and that can be.
Teacher: Did you hear his idea? Say it again please.
Cory: Aha.
Teacher: Then we can cancel *a*, and then just have one here. . .
Brian: This might work.
Teacher: Try it. You are doing a wonderful job. Don't erase, you can write it down.

All students took on the challenge of trying to do something with the initial expression; this was the first time during the teaching experiment that they had to use algebra to produce equivalent expressions to make explicit what was implicit in the initial expression. The ultimate goal was to obtain −7, which would prove their conjecture. This differs from a more traditional approach to the teaching of algebra that calls for students to "simplify" without having students pursue "simplify" as the explicit goal.

Once embarked in transforming the initial expression, students face challenges related to the use of properties such as distributivity, cancellation, and addition of like terms. This was expected and was pedagogically addressed by the teacher; teacher interventions mostly consisted of reminding students about these properties and checking students' correct/incorrect use of them.

Students in this teaching experiment adopted three strategies when producing a chain of equivalent expressions: (1) traditional *equation solving* strategy, (2) *formal derivation* strategy, and (3) the *hybrid* strategy.

None of the students used an equation solving strategy.[4] This seems reasonable since at this point in the problem solving process, the nature of the students' work was to show that for any value of the independent variable, the outcome will always be −7, as opposed to "find the value of *x*."

To classify a student's strategy as *formal derivation* (Fig. 9), −7 should not have appeared on the other side of the equal sign in the initial expression and in any of its equivalent expressions. Expressions equivalent to the initial one should be obtained by the application of properties (e.g., distributive). Note that here, in contrast with the strategy described above (i.e., equation solving) equivalent algebraic expressions are obtained, as opposed to equivalent equations.

In Problem 1, 2 out of 8 students (25%) used a *formal derivation strategy*.

To classify a student's strategy as *hybrid* (see Fig. 10) the −7 had to be kept after the equal sign in every single line and the subsequently equivalent algebraic expressions are obtained by applying properties on only one side of the equal sign, as opposed to both sides.

Six out of 9 students used a hybrid strategy, keeping the conjectured outcome (i.e., −7) on the right hand side of the equal sign. These students did not operate on the "−7" or engage it in any operations. This shows that the conjectured outcome (i.e., −7) was not used to obtain equivalent expressions; rather, it seemed to be a reminder for the students of the conjectured value of the outcome.

In sum, students faced several challenges during this stage of their work. The first challenge was to understand that something needed to be done to the initial expression to ultimately obtain −7. The second challenge was related to the correct use of properties (e.g., distributivity). Both of these challenges were overcome by students. In doing so, we were able to identify two strategies that students used in producing a chain of equivalent expressions, namely the *formal derivation* strategy and the *hybrid* strategy. All students were able to produce a chain of equivalent expressions leading to the conjectured outcome (i.e., −7); most of them used a hybrid strategy rather than a formal derivation strategy. This evidence indicates that students were able to use algebra as a tool to prove.

[4] As its name indicates, the first strategy involves equation-solving operations applied on both sides of the equal sign that keep the solution set invariant. This last feature is important because it indicates that both sides of the equal sign are engaged in an equation solving strategy. In this approach, by applying properties (e.g., associativity) a chain of equivalent equations is obtained.

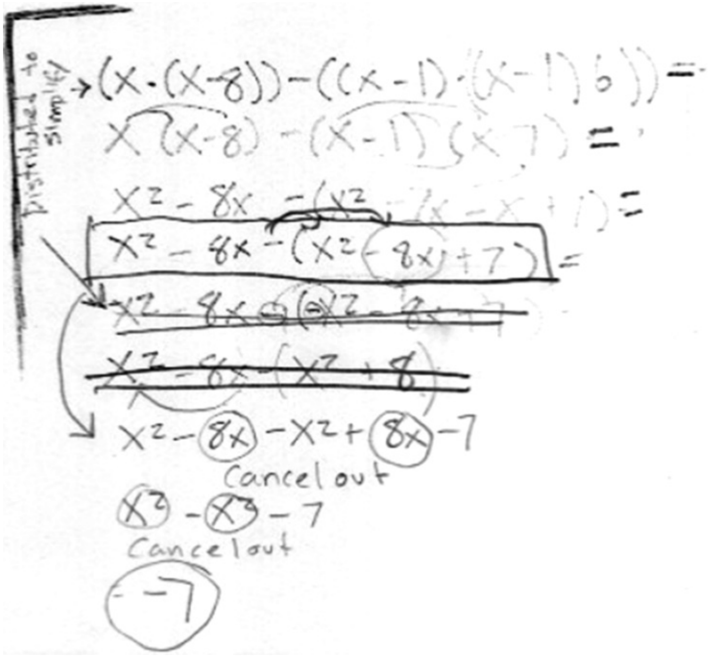


Fig. 9. Grace's formal derivation strategy.

5.2.5. Stage 2e: interpreting the final expression

All students were able to interpret the final expression in the chain of equivalent expressions either in the case of "−7 = −7" (Fig. 11) or ". . . = −7" (Figs. 12 and 13). This indicates that students were able to use algebra to prove in a meaningful way.

Moreover, students were able to connect the meaning of the initial expression [e.g., $x(x+8)-(x+1)(x+7)$] to the meaning of the last expression throughout the chain of expressions. This indicates that students were using the notion of equivalence among expressions to transfer meaning from the end to the beginning of the chain of equivalent expressions and vice versa. Some of the students added why and how that happens algebraically: namely, that the variables involved in the calculation

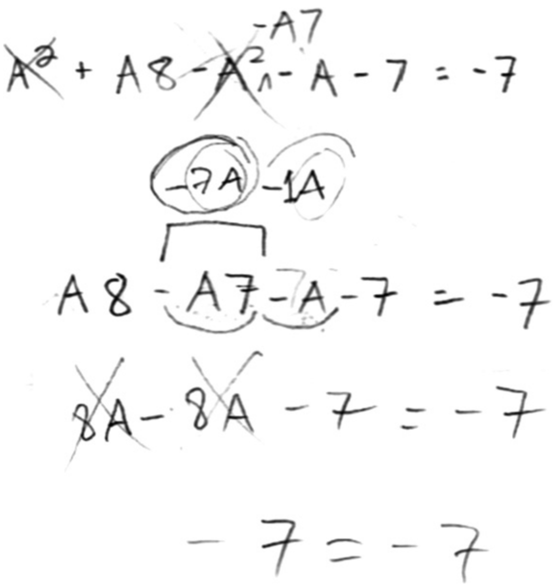


Fig. 10. Tyler's hybrid strategy.

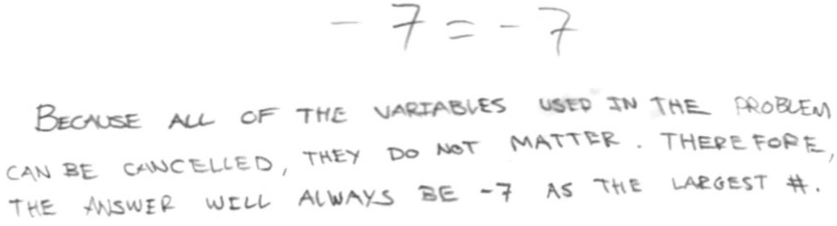


Fig. 11. Tyler's interpretation of the final expression.

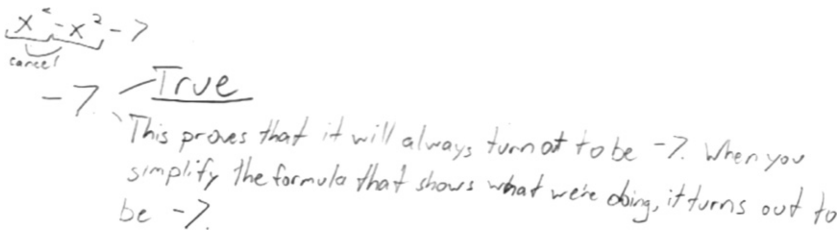


Fig. 12. Desiree's interpretation of the final expression.

they cancel each other out. The following excerpt from Tyler's group illustrates this point:

Tyler: So then since those cancel [referring to a^2 and $-a^2$].
Brian: You just need 8*a* and 8*a* minus *a*8 is just 7 minus 7.
Teacher: 8*a* minus.
Tyler: 8*a* so basically. . .
Teacher: Ok, I want you to analyze what this expression means now [referring to $-7=-7$].
Tyler: Basically we just find that minus 7 is equal to minus 7.

. . .

Teacher: The question here now is what did you find out finally?
Tyler: We know that the variables don't matter basically.
Cory: It will always equal. . .
Tyler: But no matter *what variable* we put in here it will always come out to negative 7 because we can always cancel everything out.
Tyler: So it doesn't matter.
Teacher: Ok so what I want you to do is to write a conclusion for the problem.

. . .

Cory: Ok, so what do we say?
Brian: This shows. . .
Cory: What does this show?

. . .

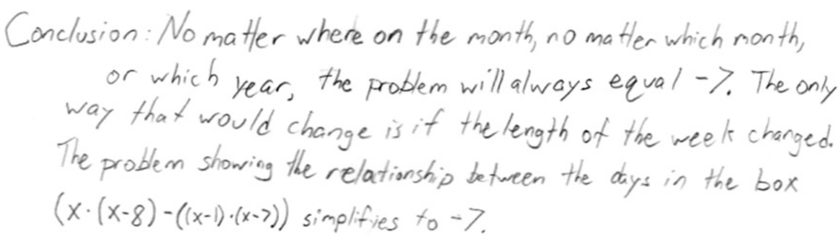


Fig. 13. Desiree's conclusion for Problem 1.

Tyler: . . . because all the variables used in the problem can be canceled.
Cory: So what's the conclusion of all this?
Brian: Shows that. . .
Tyler: Oh yeah, how does that relate back to, what's the original question? I don't even remember. . .
Cory: Let's see, what's the largest number you can find? It's always going to be negative 7 because it's always a week away.

. . .

Tyler: *Because the variables used in the problem cancel they don't matter* therefore the answer will always be 7.
Cory: Negative 7.
Cory: Because of the variables used in this problem they always cancel each other out.

A similar discussion took place in Abbie, Desiree, and Grace's group, as illustrated below by the following excerpt:

Teacher: What happened?
Desiree: It turned out to be negative 7.
Grace: So those cancel each other out [referring to x^2 and $-x^2$] and those [referring to $8x$ and $-8x$] cancel out and that left negative 7?
Teacher: Oh wow, how did you do that?
Grace: How did we get them to cancel out?
Teacher: Yeah how, what, walk me through please.
Desiree: So basically we kept distributing so we did it wrong before but it turned out that we got to $8x$ I mean, sorry. . .
Teacher: x squared.
Desiree: x squared minus $8x$ minus x squared plus $8x$ minus 7, so negative $8x$ and positive $8x$ just cancel each other out because if you're going to add $8x$ and then subtract it there's no point in having it at all. And then the same thing down here with the x squared, if you're going to add x squared and then subtract it right away there's no point in having it so it turns out it's just negative 7.
Teacher: And what does it say?
Grace: Umm.
Desiree: Well that's the formula we had for what we're doing in the calendar problem, so like how the numbers relate to each other, and we had up here what we're always going to be doing [referring to $x(x-8)-(x-1)(x-7)$] and what we're always going to be doing is gonna just turn out to be negative 7 [pointing at the last expression within the chain] because that's what it simplifies to, so it's really is always going to be negative 7.

As discussed above, the connection between the meanings of the first expression (i.e., "what we're doing") and the last expression (i.e., -7) is an indication of the transfer of meaning through the use of equivalence throughout the chain of expressions. In addition, students explained that the initial expression reduces to -7 given that all variables cancel out. The transfer of meaning along with the justification for why this happens is an indication that students were engaged in sense making when solving Problem 1.

6. Discussion and conclusion

The teaching experiment presented here is original in that it provides us with a detailed analysis of students' learning process while solving problems that involve multiple variables within a proof context in a classroom environment. Moreover, the results indicate that an integrated approach can provide students with an opportunity to meaningfully learn about aspects of both algebra and proof. Indeed, all students were able to produce an algebraic proof for a conjecture produced on their own. Specifically, all students were able to produce a conjecture about a mathematical situation involving multiple variables; in doing so, students meaningfully used algebraic notation to represent the multiple variables involved in the problem; students defined the mathematical relations among variables; and students were able to meaningfully use algebra to produce a proof, namely an algebraic proof. This proof gave them an understanding of why this particular phenomenon happens.

Acknowledging students' difficulties with multiple variables as documented by previous studies (e.g., Bardini et al., 2005; Binns, 1994; Bloedy-Vinner, 1994, 1996; Furinghetti & Paola, 1994; Sedivy, 1976; Ursini & Trigueros, 2004), and students' difficulties learning proof (e.g., Balacheff, 1982; Harel & Sowder, 1998; Healy & Hoyles, 2000; Martin & Harel, 1989), we have focused on three endeavors in this paper as described in the introduction.

First, we articulated our working hypothesis that integrating algebra and proof using algebraic proof as a vehicle may promote meaningful learning of aspects of both algebra and proof. Our approach is centered on a specific aspect of algebra: the capability to transform an expression and make explicit information that remained implicit in the initial expression. This is the connection between algebra and proof that we set out to pedagogically exploit in the design of this teaching experiment. These results constitute empirical evidence that supports our hypothesis towards an integrated approach to algebra and proof.

The second goal of this paper was to describe the design principles that guided the development of the Calendar Algebra Problems, specifically, to explain how these principles were embodied in Problem 1.

The third goal of the paper was to provide a detailed description of students' mathematical process when working on Problem 1. Students went though different stages as they worked on Problem 1. Each of these stages was defined according to the goal that directed their activities. In other words, the stages are differentiated from one another according to the goal that was orienting students' work: (1) Conjecturing and (2) Proving. Within the second stage, the following were identified as substages: (2a) Constructing a generic square, (2b) Setting up an initial expression, (2c) Generating a chain of equivalent expressions, (2d) Interpretation of the last expression, and (2e) Interpreting the final expression.

In the second stage (i.e., Proving), students meaningfully used algebra as a modeling tool (Chevallard, 1989). Specifically, students (a) used multiple variables to represent the elements involved in the problem; (b) identified and expressed the relationships among the multiple variables; and (c) represented the outcome using a complex expression involving multiple variables and several operations; this corresponds to the first two steps in Chevallard's modeling process using algebra.

In addition, using mathematical properties, students were able to produce a chain of equivalent expressions and to produce an algebraic expression that explicitly shows what they wanted to prove; this was one of the central aspects at play in our approach. Lastly, and perhaps most importantly, students were able to interpret the final algebraic expression in terms of their initial conjecture and the problem. This indicates that students meaningfully used algebra as a tool to prove when solving Problem 1.

There are two aspects of students' work that we identified as central in grounding the production of an algebraic proof. First, students produced their own conjectures as a result of their exploration of the problem; therefore, they felt certain of the plausibility of the conjecture and the conjecture made sense to them. The conjecture was the product of their own thinking; it was *their* conjecture as opposed to a conjecture provided by the teacher. This provides more evidence towards Boero et al.'s (2007) claim stating that conjecturing and proving are interrelated mathematical processes and that they should not be taught separately.

Second, students' generic 2×2-calendar-squares worked as a placeholder for the multiple variables and the mathematical relations in the problem; it provided a cognitive structure to support students' conceptualization of the problem and the production of an algebraic proof. Thus, we hypothesize that the representation of the variables and mathematical relationships played a central role in structuring students' modeling of the problem.

The study presented here is only the starting point of a much-needed systematic study of ways of integrating proof beyond the geometry curriculum in high school, specifically in more advanced algebra involving multiple variables. A replication of this study will be important in advancing our understanding of students' learning process as they work with multiple variables in a proving context. It will also contribute to a better understanding of the extent to which the difficulties students display as reported in previous research are inherent to the mathematics at stake or to the learning environments that we offer students.

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