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Multiplication Principle Problems and Permutation Problems

How many different outfits are possible when assembled from three pairs of pants and five shirts? In how many different ways can four students be arranged in a line?

These problems may seem related, but there are mathematical differences, differences that may seem clear to us as teachers but less so to students. The purpose of this article is to describe an error that students make as a result of difficulties in distinguishing these two scenarios and to consider how existing curricula and sequencing of topics may contribute to these difficulties.

The first question—regarding three pairs of pants and five shirts—relies on the multiplication principle. This principle, often referred to as the product rule or fundamental counting principle, states that if an event occurs in m ways and another event occurs independently in n ways, then the two events together can occur in $m \cdot n$ ways. If we must choose one shirt and one pair of pants and we have 3 ways to choose a shirt and 5 ways to choose pants, then we have $3 \cdot 5 = 15$ ways to choose one of each.

The second question—that of arranging four students in a line—is a permutation. The term *permutation* refers to an arrangement of objects in which the order of arrangement matters. In the case of n objects, where all n must be arranged, the number of permutations is $n! = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot 1$. For our example of arranging all 4 students, the number of permutations is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. We could also have a permutation of n objects where some number less than n must be arranged. For example, if we had 20 students, in how many ways could we arrange just 4 of them in a line? The number of permutations here is $20!/(20 - 4)! = 20!/16! = 20 \cdot 19 \cdot 18 \cdot 17$.

Problems that we refer to here as *multiplication principle problems* call for a use of the multiplication principle that does *not* result in a permutation of items from within a set. In the "shirts times pants" example, we consider pants as one set of items and shirts as a different set. We will reserve the term *permutation problems* for questions that call for an arrangement of items from within a set.

Problems arise when students overextend the strategies that they use for solving multiplication principle problems to solving permutation problems. That is, they respond to permutation problems by identifying two or more numbers and then multiplying them. Consider the problem of arranging 4 of 20 people. One possible student response might be $20 \cdot 4 = 80$. Instead of permuting items from within the set of 20 people, the student likely multiplied the number of objects of one type (20 people) by the number of objects of another type (4 spaces for people). This is analogous to a "shirts times pants" approach.

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Past research shows evidence of this type of error. For example, Fischbein and Grossman (1997) found students claiming that the number of permutations of four objects is 4 • 4 and the number of permutations of five objects is 5 • 5, multiplying the number of items by the number of positions; this type of reasoning was also described in Fischbein and Gazit (1988). Here we show additional data about the prevalence of this error, show student explanations that illuminate their thinking, and consider links to curricula and instruction.

A CURRICULAR APPROACH TO THE MULTIPLICATION PRINCIPLE AND PERMUTATIONS

Students are often introduced to the ideas and representations of combinatorics, including lists, tree diagrams, and the slot method, through multiplication principle problems. These problems appear throughout K–12 education, with specific attention to tree diagrams and lists first appearing in grade 7 in the Common Core State Standards for Mathematics (CCSSI 2010). In particular, some middle school curricula use tree diagrams and lists to illustrate problems using the multiplication principle (e.g., Lappan et al. 2006). The multiplication principle appears again in high school algebra classes. As an example, a popular Algebra 1 textbook (Carter et al. 2010) shows tree diagrams and lists applied to multiplication principle problems and then uses multiplication. However, the text then moves directly from a multiplication principle problem to a permutation problem, offering no information about how these types of problems differ and without using tree diagrams or lists for permutation problems. A similar approach is adopted in some Algebra 2 textbooks (see, e.g., Collins et al. 1997; Glencoe/McGraw-Hill 2010).

Mathematically, the multiplication principle enables a proof of the formula for permutations (Rosen 2003), but the connection requires thinking carefully about the events and the number of outcomes. When using the multiplication principle, one multiplies the number of outcomes for each event to find the total number of outcomes. For permutations, if we take our example of arranging four students, we can define the first event as putting a student in the first position, with 4 possible outcomes. A second event, putting a student in the second position, has 3 possible outcomes because there are only three students remaining. Our third event is defined similarly; and the fourth event puts a student in the fourth position, with only 1 possible outcome because there is only one student left. Although we come around again to the factorial, 4!, using the multiplication principle and choosing events carefully, neither the differences nor the similarities between problems of these two types are necessarily apparent to students.

RESULTS FROM A SMALL-SCALE STUDY

We interviewed eleven Algebra 1 students from two different classrooms (seven students from one classroom and four from the second) as part of a larger study (Caddle 2012). We interviewed all students who were willing to participate, and the interviews included two permutation questions:

1. In how many ways can you arrange these objects? [Students are shown squares labeled A, B, C, and D.]
2. If there are 10 students in an after-school club, in how many ways can the club select a president, vice president, and treasurer?

In the interviews, we observed the "shirts times pants" error described earlier. For the first question, 6 of 11 students (55%) used an incorrect multiplication principle premise; for the second question, 5 of 11 students (45%) did so. Of the 11 students, 8 (73%) used the principle incorrectly on at least one of the two questions, while 3 of the 11 students (27%) did so on both.

Question 1: Arrangement of All Objects in the Set

The correct answer to question 1—that is, 4! = 4 • 3 • 2 • 1 = 24—could have been determined using the factorial, the slot method, a tree diagram, a list, or some nontraditional strategy. **Table 1** shows more information about how the 11 students responded.

None of the students who incorrectly used the multiplication principle found a correct answer through other means. The interviewer (coauthor Caddle) asked students to try other strategies; five of the six students attempted to make a list, and one attempted to make a tree. Since their work on the lists or trees was secondary, we do not know whether the students would have been able to use the diagrams correctly if they had started with them. That is, students seemed convinced that their initial answers were correct, and when working on the list or tree, they were trying to justify that answer. For example, one student, José, moved from his initial answer of 16 into correctly beginning a list. However, he stated that it would go until he got to 16.

The four students who were able to find the correct answer all multiplied to find 4 • 3 • 2 • 1. However, three of the four were not able to use another strategy or explanation; either they applied other strategies incorrectly or relied on a "that's what my teacher told me" rationale. In fact, some of the same students who answered question 1 correctly still misapplied the multiplication principle on question 2. The fourth student who multiplied to find 4 • 3 • 2 • 1, Chris, was able to correctly start a tree diagram but

Table 1 Student Responses to Question 1

Type of Response	Correct Responses	Strategy	Number of Students
Incorrect use of multiplication principle	0 of 6 students found the correct answer	4 • 4	5 students
		2 • 4	1 student
No incorrect use	4 of 5 students found the correct answer	4 • 3 • 2 • 1	4 students
		Incomplete list	1 student

Table 2 Student Responses to Question 2

Type of Response	Correct Responses	Strategy	Number of Students
Incorrect use of multiplication principle	0 of 5 students found the correct answer	3 • 10	4 students
		1 • 1 • 1 • 10	1 student*
		10 • 9 • 8 • 7 • 6 • 5 • 4 • 3 • 2 • 1 • 3	1 student
No incorrect use	2 of 6 students found the correct answer	10 • 9 • 8	2 students
		10 • 9 • 8 • 7 • 6 • 5 • 4 • 3 • 2 • 1	1 student
		Gives a number of options for each office without multiplying	2 students
		Incomplete list of permutations of 10 students	1 student

*This student also tried 3 • 10, so these numbers do not total 5 students.

also used his own strategy, in which he examined the objects and found 6 permutations starting with A; he then multiplied the 6 by 4 to account for the different starting possibilities. Chris was the only student to correctly use multiple strategies on question 1.

Two examples illustrate the connections to multiplication principle problems and classroom experiences. One student, Donald, answered by writing "16 ways," saying that he multiplied 4 by 4 because he had learned how to do it the "fast way." He explained that instead of listing possibilities, you multiply the number of objects. When asked what to multiply the 4 objects by, he responded that "it's by 4 outcomes," but he was not sure why.

Another student, Gabriel, also stated that the answer was 16 (i.e., 4 times 4). He then made an explicit link to a multiplication principle problem: "There's a formula that she [the teacher] taught us . . . remember what she gave us about the cookies and stuff? If there's 3 types, 3 what you could put on it, and 3 drinks, you multiply 3 by 3 by 3, and you get your answer." Here, Gabriel used a specific multiplication principle problem to justify his response.

Question 2: Permutation of Some Objects from a Set

Question 2, asking for the number of permutations of 3 objects from a set of 10, has a correct answer of 720. As shown in **table 2**, only two students answered correctly, both by finding the product 10 • 9 • 8. Not surprisingly, no one reached an answer of 720 through lists or trees, but it is worth noting that no students made an accurate start using these strategies.

José exemplified the most common rationale for multiplication principle reasoning; he stated that he would multiply 3 roles by 10 kids, so there would be 30 possibilities. Gabriel used the same numbers as José but also tried a second approach. He said that the answer would be 10 and then justified this by writing out "student," "president," "vice president," and "treasurer" as his categories (see **fig. 1**). He then explained that there are 10 options for students but only 1 option each for president, vice president, and treasurer. He referred to each of the four terms as a "type" and said that you multiply all the "types" together, obtaining 10 • 1 • 1 • 1 = 10. In another incorrect application (10 • 9 • 8 • 7 • 6 • 5 • 4 • 3 • 2 • 1 • 3), David first wrote out 10 • 9 • 8 • ⋯ • 2 • 1, misapplying a strategy for permutations. He then explained that this took care of the 10 students but that then you have to multiply by 3 for the three roles. Unlike José and Gabriel, David used a permutation, but he still applied the multiplication principle to find his final answer.

These explanations show that students incorrectly applied the multiplication principle in a number of ways, all of which were based on multiplying the number of things of one type by the number of things of another type.

Connections to the Classroom

We observed both the Algebra 1 classrooms of these eleven students. Although the classrooms in this study did not use a textbook, in both classrooms the order of instruction matched the curricular materials described here, in that students were first exposed to multiplication principle problems and then to permutations.

Both teachers initially used lists and tree diagrams to solve problems such as "dresses times scarves" and "ice cream flavors times toppings." Following this, students were able to make conjectures about the multiplication principle and to justify their conjectures. The students saw the tree diagrams and lists as more onerous and saw multiplying

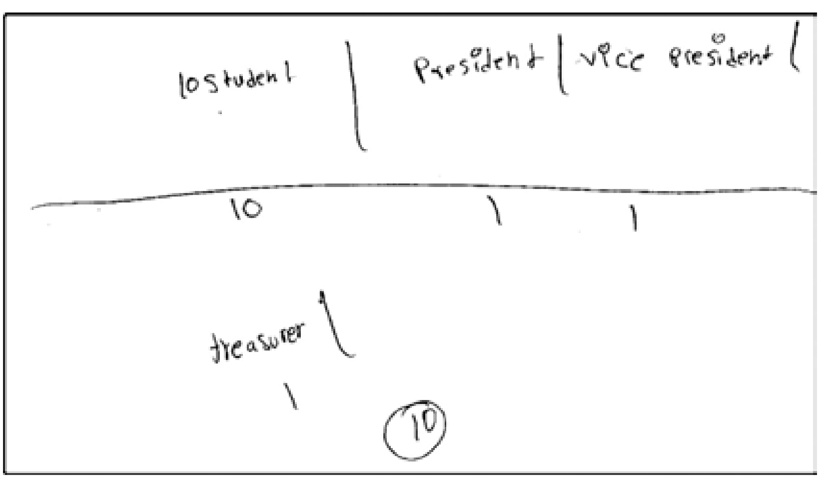

Fig. 1 Gabriel's incorrect solution for question 2 shows the number of options for different categories or types.

the number of options as a shortcut, or "fast way." However, students did not find the number of permutations using lists and trees as they had done for the multiplication principle. There was little discussion of how the permutation problems differed from the previous problems.

Past work by Batanero, Navarro-Pelayo, and Godino (1997), in their analysis of 720 written assessments, also showed that students used tree diagrams infrequently and usually tried to find a formula instead, demonstrating that, on a larger scale, students are inclined to use the most expedient strategy. Students should try to be efficient, but this analysis emphasizes the importance of considering how they might misapply their "fast ways."

IMPLICATIONS

Both the observations and the review of curricula show that a common instructional sequence relies on the implicit assumption that the multiplication principle will serve as an entry point into permutations. We argue that this common sequence has the unintended side effect of predisposing students to a misuse of the multiplication principle in permutation problems. Here we suggest classroom supports to allow students to develop an understanding of permutations.

One approach could be to introduce permutations with the same use of multiple strategies, such as tree diagrams and lists, and the same view toward conjecture and justification that was exhibited by the teachers for the multiplication principle. In past research, Fischbein and Gazit (1988) emphasized the importance of tree diagrams when introducing permutations. They carried out instruction that had students use tree diagrams before determining formulas, showing testing gains as a result although with no comparison to other instructional methods. Abramovich and Pieper (1996) suggested a recursive approach, in which students keep track of a permutation of one object, then two objects, then three objects, and so on—in essence, generating a list of the possibilities.

Because the instructional materials mentioned earlier do not demonstrate a thorough introduction to permutations, here we share one possible classroom strategy, applied to question 1, drawing from the teachers' instruction on multiplication principle problems. A tree diagram (see **fig. 2**) can be used to ask students how many possibilities there are for the first "choice," putting a letter in the first position, and so on. However, we should also use the tree to create the list of outcomes, demonstrating that the tree is a tool. In addition, although students may notice that they can multiply numbers in decreasing order, doing so in conjunction with tree construction can maintain the focus on *why* they are decreasing and potentially help students' reasoning in future problems.

Certainly, nobody wants to draw a tree diagram for the 720 possibilities of question 2, underscoring the practical limits of this strategy as well as students' desire to find a faster way. However, just starting the diagram (see **fig. 2b**) could help students reason that if they had 10 options for the first "choice" of the president, then for each of those 10 options, there would be 9 options remaining for the second "choice" of the vice president. This is akin to incorporating Chris's strategy from question 1, described earlier, in which he found 6 permutations starting with A and then multiplied the 6 by 4 to account for different starting possibilities. The partial tree diagram can help students determine a partial list of permutations and then reason about how many of those partial lists exist. By connecting tree diagrams, lists, and multiplication, we demonstrate that these are tools; they need not be stand-alone responses.

Another valuable instructional element could be to ask students to map the connections and differences between the problems and to address how to decide what constitutes an "event." In this way, students may come to an understanding of when different strategies are applicable.

Most important, curriculum designers, teacher educators, and teachers should be aware of this potential pitfall. This article shows that common instructional sequences, as suggested by curricula and observed in classrooms, do not take into account the difficulty that students have when making transitions between multiplication principle problems and permutation problems.

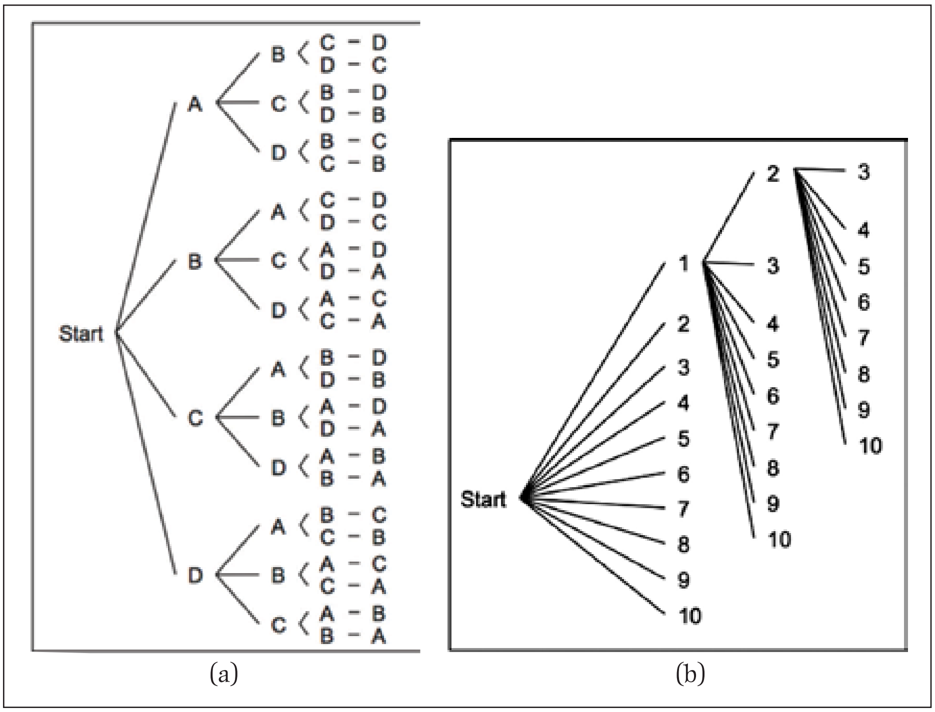

Fig. 2 Tree diagrams support reasoning about the number of permutations.

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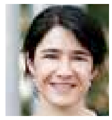

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