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Fifth graders' additive and multiplicative reasoning: Establishing connections across conceptual fields using a graph

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ABSTRACT

This paper looks at 21 fifth grade students as they discuss a linear graph in the Cartesian plane. The problem presented to students depicted a graph showing distance as a function of elapsed time for a person walking at a constant rate of 5 miles/h. The question asked students to consider how many more hours, after having already walked 4 h, would be required to reach 35 miles. To answer this question, the students needed to extend the graph that was presented, either mentally or on paper, as the axes did not go up to 7 h or 35 miles. They also needed to be able to consider not only the total number of hours to reach 35 miles, but also the interval of time after 4 h. The purpose of this paper is to consider the student responses from the viewpoint of multiplicative and additive reasoning, and specifically within Vergnaud's framework of multiplicative and additive conceptual fields and scalar and functional approaches to linear relationships (Vergnaud, 1994). The analysis shows that: some student answers cannot be classified as either scalar or functional; some students combined several kinds of approaches in their explanations; and that the representation of the problem using a graph may have facilitated responses that are different from those typically found when the representation presented is a function table.

1. Introduction

Several decades ago, Vergnaud (1982) defined and described the additive and multiplicative conceptual fields. These definitions and descriptions have proved immensely valuable for the mathematics education community, helping us both to understand the fundamental differences between additive and multiplicative approaches (Carraher, Carraher, & Schliemann, 1985; Kaput & West, 1994; Nunes, Schliemann, & Carraher, 1993; Ricco, 1982; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998; Schliemann & Magalhães, 1990; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990), as well as to design didactical interventions that might foster each kind of approach and thinking (e.g., Schliemann, Carraher, & Brizuela, 2007). However, we have also found that the approaches and ways of thinking are not as dichotomous or in opposition to each other as we once may have thought (e.g., Martinez & Brizuela, 2006; Nunes et al., 1993).

This paper examines the work of 21 fifth grade students as they look at a graph of a linear function in the Cartesian plane (from here on, "graph"). We analyze the student responses to determine how they reason about a function presented to them through a graphical representation. Specifically, this paper addresses the following research questions:

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i. When looking at a graph presented to them in a problem, can students' explanations to questions about the problem be described using the additive and multiplicative framework? Do any student explanations show evidence of thought that does not fit within the existing framework?
ii. How might these explanations be affected by the presence of the graph, as opposed to if a function table had been used?

To address these questions, we will first describe the difference between additive and multiplicative reasoning, as laid out theoretically by Vergnaud (1982). This framework will be used in examining prior research on this topic, as well as analyzing the responses of the students in this study.

We believe that it is important to consider how students use additive and multiplicative relationships, as it is only through an understanding of the multiplicative relationship that students can generalize to a function that is calculable for any value of the independent variable. In addition, studies that complement Vergnaud's framework point to the possible combination of approaches on the part of learners. It is also important to determine how the use of a graph may prompt different reasoning than another representation, such as a function table, might do. Since different representations emphasize different features of the function, as described below, we expect that the use of these representations has a profound effect on student thought (see also Brizuela & Earnest, 2008).

2. Multiplicative and additive reasoning

The primary framework for analyzing this student work is Vergnaud's theory of conceptual fields. In a general sense, a conceptual field is defined as "a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations" (Vergnaud, 1996, p. 225).

More specifically, Vergnaud (1982) outlines elements of the field of both multiplicative and additive structures. One of the key principles of these fields is that the concepts within them are not acquired quickly, simultaneously, or even within a short period of time; each field may include items which are varied enough in cognitive difficulty that years may separate understanding of different ideas. For example, Vergnaud (1994) defines the multiplicative conceptual field as containing "multiplication and division; linear and bilinear (and *n*-linear) functions; ratio, rate, fraction, and rational numbers; dimensional analysis; linear mapping and linear combinations of magnitudes" (p. 46). In contrast, the additive conceptual field that Vergnaud (1982) describes includes not only addition, subtraction, and the idea of measure, but also "time transformation, comparison relationship, displacement and abscissa on an axis, and natural and directed number" (p. 40).

Using the ideas of the additive and multiplicative fields of understanding, Vergnaud also distinguishes between scalar and functional approaches to relationships between quantities. Simply put, the scalar approach relies on successive addition to model a linear relationship. For example, the scalar method can be applied to the following problem: if it takes 30 min to write 1 page of text, then how long does it take to write 3 pages? Applying a scalar solution, we could reason, "30 min for 1 page, so 60 min for 2 pages, so 90 min for 3 pages." We have done successive additions of the number of minutes required for one page until we reached the number of pages we were seeking. Vergnaud (1994) puts this elegantly in algebraic notation, which would be as follows for our problem:

given: $f(x)=30x$ where x is the number of pages of text
the scalar solution: $f(3)=3f(1)=f(1)+f(1)+f(1)=30+30+30=90$

By contrast, the functional approach rests more directly on the multiplicative relationship between variables. The problem above could be reasoned as "30 min for 1 page, so 3 pages times 30 min per page equals 90 min." Note that this functional approach relies on a single operation on the input variable, rather than successive operations on the output variable. Algebraically, by Vergnaud's (1994) model:

given: $f(x)=30x$ where x is the number of pages of text
the functional solution: $f(3)=30(3)=90$

The functional approach is considered to be more sophisticated (Vergnaud, 1994). As Martinez and Brizuela (2006) point out, it also leads to solutions that are more easily generalizable. The scalar approach necessarily rests on knowing the previous value of the output variable, in order to use the successive addition to find the next value. This, in effect, results in a recursive function. For our example, the function could be considered as:

given: $f(x)=30x$ where x is the number of pages of text
the recursive solution: $f(3)=f(2)+30=[f(1)+30]+30=[30+30]+30=90$

However, this does not lead to the establishment of a clear functional relationship between the two different variables that is usable without having determined a previous value. For ease of computation with large values of the input variable, which would require numerous successive additions, or for producing a general functional expression for an unknown value of the input variable, the functional approach becomes important.

The scalar approach also uses or reinforces the belief that there is a countable "next" value for both the independent and dependent variables in a function. While this may, in fact, be the case for many problems with an extra-mathematical context, as students progress in mathematics, they are expected to appreciate the density of the real number line and come to understand the idea of continuous functions. In our example, it would be permissible to consider a case in which two thirds of a page of text had been written, giving $f(2/3)=30(2/3)=20$ min. In reviewing how we solved this case, the functional approach was required as there was no "previous" value of the function upon which we could rely. In fact, the ability of students to cope with a functional approach when dealing with a multiplicative function may also help with their understanding of multiplication in general. As Greer (1994) points out, multiplication may first be interpreted or calculated by students as an instance of repeated addition. However, when students move to multiplication that uses non-integer values, the technique of repeated addition is no longer viable. The difficulty of this extension of the meaning of multiplication that Greer puts forth suggests the importance of situations in which students deal with multiplication directly (i.e., in a functional way, in Vergnaud's terms), rather than treating it as repeated addition (i.e., in a scalar way, in Vergnaud's terms).

3. Previous research and use of graphs

The students described in this paper were nearing the end of a three-year intervention in which algebraic activities had been integrated into their elementary school mathematics classrooms during third, fourth, and fifth grade. This project is known as the Early Algebra (EA) project, and this paper represents a small piece of the work that this project encompasses (see also http://www.earlyalgebra.org).

Other data from the EA project has already been analyzed using the framework of scalar and functional approaches to data presented through *tables* (Martinez & Brizuela, 2006; Schliemann, Carraher, & Brizuela, 2001; Schliemann et al., 2007). This paper, in contrast, will deal with students who are working with a *graph* and a word problem presented through natural language, when no table is presented or suggested. Schliemann et al. (2001, 2007) have observed that when using two-column tables of data, with one variable in each column, third grade students preferred a scalar approach, and would often work to fill in each column independently by following the pattern of change for the individual variable, rather than considering the relationship between the columns. In order to see if students could transition to using more functional approaches, various types of activities were used. One significant change was to add an "*n*th" row to tables given to the students, so that students could consider the value of the output variable given that the input variable could be any number. This highlighted the link between the two columns. In addition, the values in the input column were changed so that they were not sequential and there were large jumps in the sequence from one row of the table to the next; this broke the within-variable pattern on which students had relied, resulting in a conflict for them. Finally, very large values for the input column were included, again with the purpose of discouraging an exclusive scalar approach on the part of the students. Some of the third grade students that Schliemann et al. (2007) report on were able to recognize and create general expressions for functional dependencies as complex as $2x–1$.

Martinez and Brizuela (2006) also looked at a case study of one of the third grade students participating in the EA project. In their analysis, they discuss the work of a student using function tables similar to those mentioned above in Schliemann et al. (2001, 2007). However, the student did not always use a scalar approach – working within the column – or a functional approach – working with a consistent operation from the input variable to the output variable. Instead, the student considered the additive relationship from the input variable to the output variable, but also looked at how the additive relationship changed. The relationship, in one of the cases, was $2x + 2$, and the student noticed that the additive difference between the variables increased by one for each increase of one in the input variable. Martinez and Brizuela characterize this approach as "*neither scalar nor functional*. At the same time, it is *both* scalar and functional" (p. 291, italics in original). They do conclude that ultimately the student may find it helpful to adopt a more conventional functional approach, in order to deal with the problem of generalizing a relationship to an unknown value for the input variable, as discussed above. Martinez and Brizuela's (2006) study is relevant to the study reported in this paper because of the need to look beyond the traditional definitions of scalar and functional approaches and the possibility that students' explanations may not neatly fit into these two categories.

These previous analyses have focused on approaches to functional relationships with values represented through function tables. However, a graph provides a different representation for the same function, highlighting other aspects of the functional relationship (e.g., Brizuela & Earnest, 2008; Moschkovich, Schoenfeld, & Arcavi, 1993). While adult, expert users of functions can easily find the connections between a table and graph for the same function, these are very different for algebra learners, leading them so far as to treat each representation of the same function as corresponding to different functions. Mathematically, graphs can give a picture of *all* values of a function over a limited domain, assuming that the function is defined for a continuous set of real input values. By contrast, a table can only show a limited number of discrete input values and their corresponding function output values. However, when using a table that includes an unknown, *n*, as an input value (e.g., Martinez & Brizuela, 2006; Schliemann et al., 2007), the table can become a location to consider the generalized function using the algebraic expression. The graph gives us a more complete view of the function over a limited domain, as compared to discrete values in a table, while the algebraic expression, even when considered as part of the table, gives all the information about the function for any input value; for the purposes of this discussion, any real number. It is because of these differences that it is worthwhile to consider scalar and functional approaches as used in concert with graphs, without limiting ourselves to function tables (Brenner et al., 1997; Brizuela & Earnest, 2008).

For the learner, the mathematical differences between representations, as described above, are only part of the picture. The superficial appearance of a graph is clearly quite different from that of a table, and learners must be able to interpret the construction of the graph and how information is represented on it. A point on the Cartesian plane stands for an ordered pair of values, and for a function, it can tell us a particular input value and the corresponding output value, all through a single point. This rather amazing fact, which we take for granted in our adult lives, is a concept that is not necessarily intuitive to learners of algebra. As we look at student thinking, then, we should remember that each learner is doing complex mathematical work as he or she interprets a graphical representation.

This consideration of the existing work on elementary students using scalar and functional approaches will guide the analysis below. From a quantitative point of view, the frequency of each type of approach will be addressed. From a qualitative point of view, we will consider the work that the students did in this problem, working with a graph, and consider the structure of this graph-based question as opposed to other questions that have focused on tables.

4. Methodology

The student work described in this paper is part of a larger longitudinal study of algebraic reasoning in the elementary school classroom known as the EA project (Schliemann et al., 2007). This project focuses on the idea that algebraic reasoning can support and enhance understanding of arithmetic concepts, and that introducing algebraic concepts in the early grades can enhance students' mathematical understanding in the long term. A functional approach to algebra is used, in which the emphasis of instruction is not on symbolic manipulation, but rather on generalized relationships and the representations of those relationships through different means: tables, natural language, graphs, and algebraic expressions.

The students described in this paper received instruction as part of the EA project for three years, while they were in third, fourth, and fifth grades. The students were in two different classrooms in the same diverse, urban public school in Boston, Massachusetts. For the first two years of the intervention, the research team implemented two one hour lessons per week in each classroom, as well as two half-hour homework reviews for the work related to each lesson (50 lessons in third grade and 36 lessons in fourth grade). During the fifth grade year, each class had one 90-min lesson per week, and one 45-min homework review (18 lessons in total during the fifth grade).

Over the course of the three year intervention, students worked to establish relationships between generalized quantities (Schliemann et al., 2007); to use number lines for representing relative magnitude; to use the Cartesian plane to represent relationships; to use function tables (Schliemann et al., 2001, 2007); to use notation for unknown quantities; to compare relationships; and to use symbolic manipulation to solve equations.

During the third grade year, students were first exposed to the idea of representing a linear function on the Cartesian plane. The first introduction used a "human graph" where students represented a point on the line of a linear function by standing on that point on the floor. The discussion began by using two parallel number lines, each representing a different variable, with a dependent relationship between the two. Eventually, one line was rotated 90° to be perpendicular to the other; as such, the Cartesian plane was formed (for other descriptions of this activity, see also Schliemann & Carraher, 2002; Schliemann, Carraher, & Caddle, 2008).

From these beginnings, students continued to work on graphs, including activities where they created their own graphs. For example, students might first construct a table of ordered pairs of values that aligned with a particular linear function, and then use this table to plot points on the Cartesian plane. They also were exposed to graphs, both linear and non-linear, that had already been produced, and they discussed the interpretation of these as a class or in small groups.

At the end of their fifth grade year, as the conclusion of the intervention neared, students completed a written assessment. Following the assessment, each student was interviewed by a member of the research team. During the interview, three of the questions from the assessment were discussed with the student. The goal of the individual interview was to elicit student descriptions of their mathematical thinking. During the interview, they reflected on their previous written answers, and also considered questions from the interviewer about the mathematics problems. Fig. 1 shows one of the questions that was discussed with students, regarding a graph representing the distance José walked as a function of time. During the interview, the students were asked about parts (1) and (4), as shown in Fig. 1, but they were also asked a new question about the same graph and story; this question had not been part of the written assessment:

> Beginning at 4 h, how much more time would José need to reach a total of 35 miles if he continues at the same pace? How do you know?

This additional question during the interview required students to consider and discuss an aspect of the problem that they were being asked about for the first time; this new question is the focus of this paper. [For a different analysis of other questions about this graph, see also Schliemann et al., 2008.]

All interviews were videotaped and transcribed in preparation for analysis. For each interview, a coder began by recording the initial answer that each student gave to the question, and the final answer that the student gave after he or she discussed the question with the interviewer. The student explanations were then coded as well, with multiple codes being applied to a single student if the student provided more than one type of explanation.

The initial categorization of student explanations was undertaken *a priori*, yielding the categories identified as 1, 3, and 4 in Table 3. This was based on the expectation of observing either the scalar progression of adding 5 to the previous number of miles (Category 1), or using the multiplicative – and functional – relationship of 5 miles/h in conjunction with the number of hours (Category 3), or else a combination of these (Category 4). This coding scheme was suggested by the aforementioned work on scalar and functional approaches, and the additive and multiplicative conceptual fields (Martinez & Brizuela, 2006; Vergnaud, 1994). However, as specifically stated by Vergnaud (1994), a precise mathematical analysis of student thought cannot be fully achieved until we are actually confronted with specific examples of student thought. As a result, Category

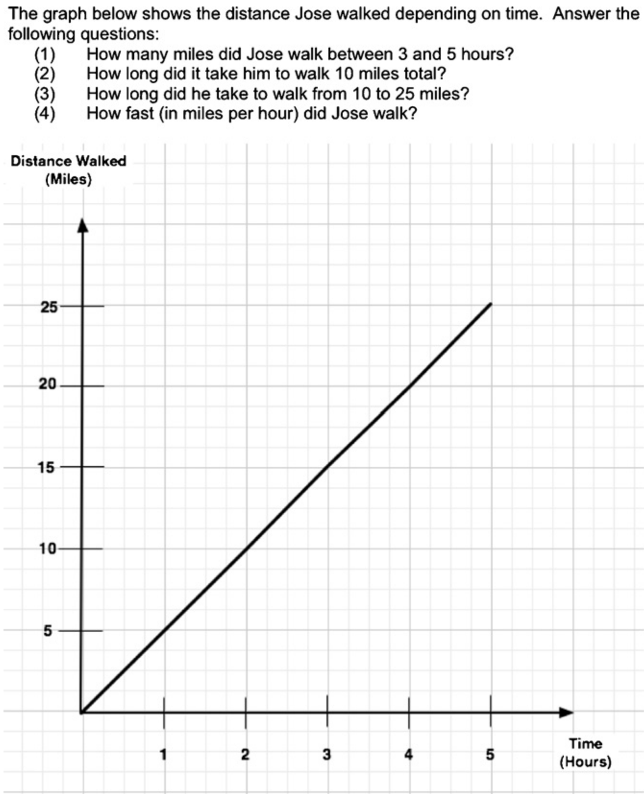

Fig. 1. The written assessment question, as well as the graph given to the students.

2 was established *a posteriori*. This category captures explanations in which the student adds an interval of more than one "increment" as defined by the scale on the graph (>1 h to count of hours and/or >5 miles to count of miles). Note that since some students used more than one type of explanation, and are included in Category 4 as a result, they are also included in the totals for Categories 1 through 3, resulting in a total percentage that is greater than 100%. This was done so that the relative frequency of the first three categories can be easily seen; otherwise, the details of students using more than one type of explanation would be obscured.

5. Results

Of the 21 students who were asked this new question during the interview, only three students (14%) gave the expected answer, 3 h, as their first response; this is shown in Table 1. Notice that nine students (43%) initially gave an answer of 7 h. In fact, as seen in Table 1, 7 h was by far the most common answer given. This response is the correct answer to the question of how many hours *in total* would be required for Jose to walk 35 miles, rather than the number of *additional hours* required after the first four had already elapsed. In this case, it is reasonable to consider that students made accurate mathematical assessments of the length of time, but neglected to attend to the aspect of the problem that asked them to only consider a portion of the total amount of time. This interpretation is in alignment with previous work that found students inclined to consider the full value of the heights of two people, rather than the value of the difference between their heights (see Carraher, Schliemann, Brizuela, & Earnest, 2006).

Nineteen students (90%) ultimately gave their answer as 3 h after discussion with the interviewer, as shown in Table 2. The discussion was at times extensive, and at times a brief restatement of the question. The most common restatement by

Table 1
Frequency of students' initial answers.

Initial answer	Number of students	% (N = 21)
2 h	3	14%
3 h	3	14%
4 h	4	19%
6 h	2	10%
7 h	9	43%

Table 2
Summary of final answers in conjunction with students' first responses.

Final answer	Initial answer	Number of students	% (N = 21)
3 *h*	2 h	3	14%
	3 h	3	14%
	4 h	3	14%
	6 h	2	10%
	7 h	8	38%
4 *h*	4 h	1	5%
7 *h*	7 h	1	5%

Table 3
Classification of students' explanations.

Cat.	Explanation characteristics	Number of students	% (N = 21)
1	Counts up by steps of 5 miles at a time	13	62%
2	Adds an interval of more than one "increment" as defined by the scale on the graph (>1 h to count of hours and/or >5 miles to count of miles)	10	48%
3	Uses multiplicative relationship (7 × 5 or 3 × 5)	7	33%
4	Students who use more than one explanation (i.e., a combination of the above explanations)	8	38%

the interviewer was in response to a student answer of "7 h," as described above. In this case, the interviewer often gave a prompt such as, "But after four hours, how many *more*?" For some students who gave initial answers of 2, 4, or 6 h, a longer discussion ensued. At no point were students given the answer of 3 h; rather, they were encouraged to look again at the graph and to consider what would happen as José continued to walk.

In addition to the numeric answers given by the students, we can also examine the explanations or reasoning that they expressed to the interviewer. Table 3 below shows the categories of student answers; the categories are then described more fully. This analysis is undertaken to allow us to look more deeply at the justifications the students provide. In doing so, we can attempt to gauge if the students used scalar or functional reasoning to answer the question.

5.1. Category 1 – Counts up by steps of 5 miles at a time

Students who gave explanations in this category verbally stated each 5 mile increment that led to 35 miles. Some of the students appeared to use a "counting up" strategy, in which they counted up by five from either 0 or 4 h, with the answer being the number of times they had incremented their count. The "counting up" strategy has been described and defined by Carpenter and Moser (1984) as one in which students begin at the starting point they perceive and continue to increment their count until they reach their numeric destination. This category is considered to be an additive strategy and is consistent with Vergnaud's scalar approach, as the students express that they are moving through values for one variable based on the previous value for said variable that they had considered. While some students stated aloud only the number of miles, other students described the additive relationship for both time and the additive relationship for distance, in parallel. In this way, they consider the ordered pairs of (*time*, *distance*) as they progress through them. In this way, the approach of some of these students mirrors the approach of street sellers described by Nunes et al. (1993). The vendors used successive addition to calculate the price for multiple items; in doing so, they maintained the link between the independent variable – the number of items – and the dependent variable – the cost. In the study that is the focus of this paper, we can show how a student, Benilde, gives an explanation that is typical of students in this group:[1]

B: He'll need 3 more hours.
EA: He'll need 3 more hours, how do you know that?
B: Because 4 plus . . . because if the 4 is 20 miles and 5 is 25 and um . . . if it was 6 then it would have been 30. Then 7 is 35 miles.

5.2. Category 2 – Adds an interval of more than one "increment" as defined by the scale on the graph (>1 h to count of hours and/or >5 miles to count of miles)

Students giving explanations in this category still used an additive relationship for distance, and/or an additive relationship for time. However, these explanations differed from those in Category 1 because they did not express, or count through, each individual increment. By increment, we mean the units of time or distance that are made explicit when looking at

[1] In all transcript examples, EA stands for the research team (Early Algebra) interviewer.

the graph. Therefore, for this particular graphical representation, an increment of time is 1 h, and an increment of distance is 5 miles. Thus, students using Category 2 explanations referred to a length of time greater than 1 h, and/or a distance of more than 5 miles. They still used a scalar explanation; however, their use of segments longer than the standard increment suggests a classification notably different from Category 1. One of the students interviewed, Alex, gives an explanation that illustrates this:

A: Because at 5 hours, he walked 25 miles, so like since every hour he walks 5 miles, I added 10 to this and got 35 (pointing to distance axis), and added 2 to this and got 7 (pointing to time axis).
EA: Good. And then you added one more because it was from four.
A: (Nods.)

5.3. *Category 3 – Uses multiplicative relationship (7 × 5 or 3 × 5)*

Students in this category gave explanations that explicitly used the multiplicative relationship between time and distance. Since this relies on the relationship between variables, without employing a recursive technique that relies on within-variable progression, this is similar to the functional approach described by Vergnaud. Some students initially calculated the whole time to reach 35 miles, stating that 7 times 5 is 35. Others addressed the problem by first determining that José needed to walk 15 miles, and then used the multiplicative relationship of 3 times 5 to find 15 miles. In either case, though, the student used the explanation of multiplying the number of hours by the rate of 5 miles/h. Dayshawn clarifies this for us:

D: 3 more hours.
EA: How do you know?
D: Cause I did. . . I did, 3 times 15 and I know that every hour is. . . every hour he walks 5 miles so if you do 5 times 3 that would give you 3 hours and that would be 15 and then you do 20 plus 15 and then you do 35.

5.4. *Category 4 – Students who use more than one explanation (i.e., a combination of the above explanations)*

This category is a secondary classification and is intended to give an indication of how many students expressed ideas from more than one of the three prior categories. For example, consider this explanation by Brianna:

B: Goes up by 5 so at 4 that's 20 so he needs 15, he's going to end at 7 hours. It's 7 hours.
EA: Okay, so he's going to end at 7 hours. But my question is, if he starts at 4 hours, how much more time would he have to continue walking for?
B: Three.
EA: How do you know that?
B: 7– if you look at 5– 5 hours twenty five, 6 hours is 30, 7 hours is 35. 4 plus 3 is seven.

Brianna initially tells us about the multiplicative relationship, saying that it "goes up by 5 so at 4 that's 20". This seems to refer to the functional relationship between time and distance and is consistent with the Category 3 responses. However, as she continues to explain, she uses the additive relationship and the ordered pairs of (*time*, *distance*) that characterized the Category 1 responses. As a result, Brianna is included in this category as having used more than one type of explanation.

6. Discussion

Many of the students clearly showed us, through their explanations provided during the interviews, whether they were using additive or multiplicative relationships to explain their answers. For those students providing Category 1, additive or "counting up" explanations, these descriptions fit closely with the scalar approach described by Vergnaud (1994). In fact, although the students are dealing with a graph, and not a table, their explanations are parallel to those often used by students in dealing with function tables, when they are using a scalar approach to increment the dependent variable, rather than considering the mathematical relationship between the independent and dependent variables. An illustration of this approach is shown in Fig. 2; students have also been seen to use this approach when working with tables as part of the EA project (see Martinez & Brizuela, 2006; Schliemann et al., 2001, 2007).

It is worth noting that, within these Category 1 responses, the students do not necessarily ignore the independent variable. However, their calculations do not depend on the relationship between hours and distance. As the interview transcript from Benilde (above) shows, the students may be cognizant of the number of hours that corresponds to the distance, and some of the students explicitly reference the number of hours that corresponds to each value for distance. This is why we have referred to these students as dealing with ordered pairs of values, and also connected this to the same tendency found with Brazilian street vendors, as described by Nunes et al. (1993). As the street vendors maintained a verbal link as they increased both quantity of items sold and price, our student Benilde maintained a connection between hours and distance even while using a scalar approach to calculate each incremental number of miles.

Number of hours	Distance walked
1	5
2	10
3	15
4	20
5	25
6	30
7	35

Fig. 2. An illustration of the scalar approach on a function table for José's problem.

The students giving Category 3 explanations explicitly use the multiplicative relationship between time and distance. These explanations fit the functional approach Vergnaud (1994) outlines. That is, there is a clear use of the dependent relationship. Rather than consider the incremental number of miles needed, students operate on the number of hours using the functional relationship of 5 miles/h. This could be considered a more useful long-term approach, as it relies on a generalized relationship. This may make it more straightforward for students to be able to state the generalized relationship in terms of a variable for an unknown number of hours (Martinez & Brizuela, 2006).

As noted earlier, responses in Category 1 and Category 3 were anticipated due to the literature on conceptual fields, as well as previous analyses of EA project data. What, then, of Category 2? The explanations defy easy categorization as scalar or functional. The approaches used by the children were additive, as described above in conjunction with Category 1, but the additive logic used increments that were greater than the obvious ones; a length of time greater than 1 h, and/or a distance of more than 5 miles. In previous work, Martinez and Brizuela (2006) have described a "hybrid" approach used by third grade students in the EA project who were working with tables, and they also discussed other examples of mixed approaches from the literature (e.g., Nemirovsky, 1996). It is possible that Category 2 corresponds to a hybrid approach as well, this time in reference to students' work with a graph.

However, our analysis of this particular problem indicates that the Category 2 explanations (1) do not show evidence of the type of hybrid approach described by Martinez and Brizuela (2006); (2) do use a combination of scalar and/or functional approaches, and (3) do not explicitly provide signs of the particular approach used. That is, students in this category show some evidence of a scalar approach, as they use an additive, recursive process to find new values within the same variable. However, since they use increments that are greater than those specified by the problem statement, we may assume that they have mentally calculated the values for these longer intervals. It is possible that the approach for finding longer intervals may be functional; however, the student explanations are not detailed or explicit enough for us to be able to confidently infer whether their approach is functional in some way or not.

In order to justify the existence of Category 2, as an illustrative example we will look again at Alex, describing how he got from 5 to 7 h:

A: Because at 5 hours, he walked 25 miles, so like since every hour he walks 5 miles, I added 10 to this and got 35 (pointing to distance axis), and added 2 to this and got 7 (pointing to time axis).

Alex refers specifically to addition; he is adding to the number of miles at his start point (25 miles) to reach his end point (35 miles). At the same time, he adds 2 h to 5 h to get an answer of 7 h. However, he does not explicitly add 5 miles successively. Instead, he adds 10 miles. In order to have obtained the 10 miles and the 2 h to add, we hypothesize that one of two possibilities exists. First, he may have silently or implicitly incremented from 25 to 30 to 35, using the additive techniques of Category 1 answers, and thus determining that 2 h would be needed. Then, knowing that he ended with a result of 10 miles and 2 h, he used this in his explanation. This would allow us to classify Alex's approach as scalar.

Alternatively, he may have first determined that he needed 10 miles, and then used the multiplicative relationship of 5 miles/h to arrive at a time of 2 h. Again, he could then use these in his explanation. If Alex were to have used this approach, he would be relying on the functional relationship between time and distance, and, as such, he would be using Vergnaud's functional approach.

Alex's words allow for reasonable arguments to be made that he may be using either a scalar or a functional approach. With the students in Category 2, then, one of the features that differentiates them is that we cannot be certain of some details of their methods. Their explanations do not make explicit, to us, how they arrived at the longer intervals they can then manipulate. Their reasoning is not necessarily different from that which could be classified using the additive and multiplicative framework; however, it cannot be parsed into these two dichotomous categories. This implies that students may move between scalar and functional thinking with more fluidity than we can assume just from students who give separate, freestanding explanations using either scalar or functional approaches at different times. Moreover, we also need to consider the possibility that this fluidity may be highlighted in our study due to the fact that we are exploring their reasoning through a graphical representation, instead of the tabular representation typically used.

This uncertainty is a reminder that all of our analysis and discussion is necessarily based on what the students express to us out loud. Attempts are made to interpret the student thought that underlies their statements and to hear the sense and meaning in the student explanations. However, what is not expressed explicitly can only be hypothesized.

In addition, for all categories of explanation, students may be using a justification that may or may not have assisted them in finding the answer. For example, some of the students first gave their answer, a number of hours, and then gave their explanations upon being asked for them by the interviewer. While it is likely that students would reflect on what they had just done mentally to provide the mathematical response, it is essential that we remember that they are providing an explanation after the fact; we are not following their cognitive processes as they take place.

This can be seen for students who give explanations in multiple categories. Witness the example of a student, Deshawn:

D: About 6. . . I mean 7.
EA: How do you know it's 7?
D: So it's 5 hours it's like 25 and 6 hours is going to be right here or right here, say right here and it's going to be 30, that's going to be 30 and say this is 35 right there and that would be just a few more, just a few more! And that since the 6 is right there, the 7 and that's like 5 times 7.

Deshawn walks us through the solution by pointing out each corresponding pair of time and distance values. He counts up through 25, 30, and 35 miles, with the proper number of hours, describing a scalar approach to the problem. However, on concluding that the walk would be 7 h, he caps his explanation by telling us that 35 is 5 times 7, referencing the functional relationship based on the number of hours. From this, we might conclude that Deshawn understands both the scalar approach and the functional approach, as he is able to use both in his explanation. We do not know, though, which approach he used mentally to reach his initial answer, or if he might have used both methods, one to check the other.

Additionally, this is a reminder that the explanation given by any student does not necessarily constitute the only explanation of which the student is capable. In this discussion, the functional approach is considered more sophisticated (Vergnaud, 1994). It also leads more directly to generalizable functional expressions (Martinez & Brizuela, 2006). However, we are not assuming that because a student does not give a functional explanation that this student does not understand the functional relationship between time and distance. The kinds of explanations that students provide are dependent on the situation as well as the representations with which the student is interacting. Thus, a particular question or focus at a particular moment in the interview may elicit a response that may be more scalar in nature, and a different question or focus may elicit a response that may be more functional. Different representations of the same function (a table, a graph, algebraic symbolic notation) may also elicit different approaches and explanations. This is why the consideration of Category 4 is so relevant in this study: over a third of the students provided more than one kind of explanation, illustrating the above points.

7. Conclusion

One unanswered question in this analysis is how the distribution of students using a scalar approach, a functional approach, or both was affected by the question's focus on a graph. For example, we do not know if the same students, at the same point in their mathematical growth, would have used a functional approach more often if the data had been presented in a table.

Previous work has shown that elementary students are capable of using a functional approach (Schliemann et al., 2001, 2007). Analysis in this paper shows that at least some students of this age use a functional approach when working with a graph. However, we have more evidence to show the development and encouragement of the functional approach in work with tables. For example, as previously mentioned, students were likely to adopt a functional approach of working across the columns of a table when the table was structured to interfere with a scalar approach. That is, the input values were not consecutive, large numbers were used for the input values, and ultimately "*n*" was used as an input (Schliemann et al., 2001, 2007).

In our work here with the graph, no such structure exists that would confer a significant advantage to the students using a functional approach. Both additive and multiplicative reasoning, or a combination thereof, enabled students to find a solution quickly for the relatively small value of 7 h. A possible future approach that could be adopted in our work with graphs in elementary school classrooms could be to structure a question to diminish the ease of the scalar approach on a graph, as has already been done with tables. A very large value for the independent variable, the number of hours, could be offered. Another strategy could be to use large or unusual explicit units, or "increments," on one of the scales so that performing scalar calculations becomes cumbersome. For example, if José walked, or perhaps rode his bike, at 13 miles/h, it would be less convenient for students to successively add 13 than it was for them to do so with five.

Alternatively, students could be asked to generalize to the case of "*n*" hours. While it is easier to show "*n*" hours in a table than it is on the axis of a graph, students could be asked to consider or imagine this case. It is possible that a structure such as this would create for the students the need for a functional approach.

As another possible consequence of the ease of use of either the additive or the multiplicative approach in this particular problem, while some students explicitly demonstrated their understanding of the functional relationship between time and distance, others may not have seen this as necessary or beneficial. In fact, we can even speculate that perhaps some students used a Category 1 explanation in their verbal response, adding on by 5 miles in their explanations, because they felt that this was the simplest and clearest way to justify or prove their answer to their interviewer.

While this is simply speculation, it does show us the strength of the discourse in the interviews where students used more than one method or type of explanation. In referring to both scalar and functional approaches in their explanations, the students make it clear to us that their understanding of the relationship between time and distance is strong and flexible. They are able to prove their answers using more than one method. Additionally, the use of the graph in both fostering and exploring children's functional thinking is crucial, as opposed to an exclusive focus on a tabular representation. We suggest that, for a future study, we should interview students with the same educational experience about the same function, presented in the same context, but providing half of the students with information in a table and half with information in a graph. This would illuminate further how the use of different representations facilitates or impedes functional thinking.

This paper does not make claims about the extent to which the students' explanations were affected by the intervention. We can say with certainty that the students in the intervention group had been exposed to function tables, algebraic notation, the Cartesian plane, and graphs of linear functions, as described above. This exposure gave them the foundation to consider the algebraic tasks that are analyzed here. We conjecture that many students in the fifth grade would not be comfortable enough with representations of linear functions to interpret the information in the graph and provide answers to the questions, although we have been able to show here that this type of work is within the reach of fifth grade students. We do not know, at this time, how students outside the intervention would answer the question that is discussed here and to what extent their reasoning would rest on multiplicative or additive structures. As a result, we make no claims here about the impact of the Early Algebra instruction. [For claims related to the impact of the intervention, see Schliemann, Carraher, and Brizuela (in press).]

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References

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Durán, R., Reed, B. S., et al. (1997). Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal*, *34*(4), 663–689.

Brizuela, B. M., & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the "best deal" problem. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 273–301). Mahwah, NJ: Lawrence Erlbaum Associates/Taylor & Francis.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, *15*, 179–202.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology*, *3*, 21–29.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, *37*(2), 87–115.

Greer, B. (1994). Extending the meaning of multiplication and division. In G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 61–85). Albany, NY: State University of New York Press.

Kaput, J., & West, M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 237–287). Albany, NY: State University of New York Press.

Martinez, M., & Brizuela, B. M. (2006). A third grader's way of thinking about linear function tables. *Journal of Mathematical Behavior*, *25*, 285–298.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69–100). Hillsdale, NJ: Lawrence Erlbaum Associates/Taylor & Francis.

Nemirovsky, R. (1996). A functional approach to algebra: Two issues that emerge. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 295–313). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics*. Cambridge: Cambridge University Press.

Ricco, G. (1982). Les premieres acquisitions de la notion de fonction lineaire chez l'enfant de 7 . . . 11 ans. *Educational Studies in Mathematics*, *13*, 289–327.

Schliemann, A. D., Araujo, C., Cassundé, M. A., Macedo, S., & Nicéas, L. (1998). Multiplicative commutativity in school children and street sellers. *Journal for Research in Mathematics Education*, *29*(4), 422–435.

Schliemann, A. D., & Carraher, D. W. (1992). Proportional reasoning in and out of school. In P. Light, & G. Butterworth (Eds.), *Context and cognition* (pp. 47–73). Harvester-Wheatsheaf: Hemel Hempstead.

Schliemann, A. D., & Carraher, D. W. (2002). The evolution of mathematical understanding: Everyday versus idealized reasoning. *Developmental Review*, *22*(2), 242–266.

Schliemann, A. D., Carraher, D., & Brizuela, B. M. (2001). When tables become function tables. In M. van der H.-P. (Ed.), *Proceedings of the 25th conference of the International Group for the PME* vol. 4, (pp. 145–152). Utrecht, The Netherlands: Freudenthal Institute.

Schliemann, A. D., Carraher, D., & Brizuela, B. M. (2007). *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice.* Mahwah, NJ: Lawrence Erlbaum Associates/Taylor & Francis.

Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (in press). Algebra in elementary school and its impact on middle school learning. Recherches en Didactique des Mathématiques, Paris, France.

Schliemann, A. D., Carraher, D. W., & Caddle, M. (2008). *From number lines to intervals in the Cartesian space*. Paper presented at the annual meeting of the American Educational Research Association, New York, NY.

Schliemann, A. D., & Magalhães, V. P. (1990). Proportional reasoning: From shopping, to kitchens, laboratories, and, hopefully, schools. In G. Booker, P. Cobb, & T. De Mendicuti (Eds.), *Proceedings of the XIV PME Conference* Vol. III, (pp. 67–73). Mexico: Oaxtepec.

Schliemann, A. D., & Nunes, T. (1990). A situated schema of proportionality. *British Journal of Developmental Psychology*, *8*, 259-268

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39–59). Hillsdale, NJ: Lawrence Erlbaum Associates/Taylor & Francis.

Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel, & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 41–59). Albany, NY: State University of New York Press.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 219–239). Hillsdale, NJ: Lawrence Erlbaum Associates/Taylor & Francis.