Pergamon

Journal of Mathematical Behavior
20 (2002) 309–319

Mathematical Behavior

Additive relations and function tables

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Abstract

We present work with a second grade classroom where we carried out a teaching experiment that attempted to bring out the algebraic character of arithmetic. In this paper, we specifically illustrate our work with the second graders on additive relations, through the children's work with function tables. We explore the different ways in which the children represented the information of a problem in the form of a self-designed function table. We argue that the choices children make about the kind of information to represent or not, as well as the way in which they constructed their tables, highlight some of the issues that children may find relevant in their construction of function tables. This open-ended format pointed to how they were understanding and appropriating tables into their thinking about additive relations.

Keywords: Function tables; Data tables; Additive relations; Early mathematics education; Algebra; Mathematical representations

1. Introduction

Although tables are an integral part of the mathematics curriculum, we know surprisingly little about how this tool lends itself to the understanding of functions, particularly of additive functions. The same cannot be said of other tools related to functions, such as graphs, where there has been considerable research describing children's understanding and construction of them (e.g., diSessa, Hammer, Sherin, & Kolpakowski, 1991; Leinhardt, Zaslavsky, & Stein,

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PII: S0732-3123(02)00076-7

1990; Nemirovsky, Tierney, & Wright, 1998; Tierney & Nemirovsky, 1995), or other types of diagrams (e.g., Sellke, Behr, & Voelker, 1991; Simon & Stimpson, 1988). What is the nature of children's understanding of tables? How does their understanding evolve? What do children learn about functions when working with tables? And what issues might educators and curriculum developers need to take into account when designing learning activities that employ tables?

Our work stands in contrast to research carried out up to this moment. Sellke and his colleagues view the use of "linking tables" or diagrams as "overrid[ing] the influence of the intuitive models and the numerical constraints associated with them, overrid[ing] the inaccurate conceptions many students have of the two operations, reflect[ing] multiplicative relationships between the quantities, remain[ing] true to the semantic meaning of the problem, and [being] effective regardless of the numerical characteristics" (Sellke et al., 1991, p. 31). Simon suggests that concrete diagrams are helpful for children to refer to when "confusions arise with the abstractions" of the algebraic problems they are confronted with (Simon & Stimpson, 1988, p. 140). Streefland (1985) has found that "the ratio table is a permanent recording of proportion as an equivalence relation, and in this way contributes to acquiring the correct concept" and "serves the progression in schematizing (p. 91)." These researchers have focused on the use of tables to enhance or guide students' understanding of functions. Our focus has been that of uncovering the understandings already present by analyzing the original self-designed tables constructed by young children. This psychogenetic approach to children's representations is similar to that carried out in the area of literacy by Ferreiro (e.g., Ferreiro, 1988; Ferreiro & Teberosky, 1979) and in the area of musical notations by Bamberger (1988).

In our classroom research, we have noticed that children tend to construct tables that differ significantly from the conventional tables they are shown by us, their teachers. This has suggested to us that children's re-construction of tables can inform us about how they are working tabular representations into their thinking — their thinking in general and about functions in particular. A case in point is that of a child named Joey. When asked to construct a table to display the available data we had been working with in a word problem, he began to peek at the printed table on the last page of his handout. As he re-constructed the table, he intermittently flipped to the end page to verify his work and move to the next step. Fig. 1 shows the table he had used as a model.

Fig. 2 shows Joey's representation of the table. At first glance, he simply transposed the table, organizing the columns by days (even though it was organized by children's names in the original table) and the rows by children. However, he did not place the children's names only once at the beginning of each row. Instead, Joey seemed to find it necessary to write the child's initials in each one of its respective cells. In other words, in Joey's re-construction, Jessica's initial recurs in each cell of row 1; Daniel's initial recurs in each cell of row 2; and Leslie's initial recurs in each cell of row 3.

In the present paper, we explore children's self-designed function tables with the goal of: (a) learning about what they make explicity in the construction of a function table; (b) learning more about children's understanding of additive functions, as reflected in their self-designed tables.

	Jessica	Daniel	Leslie
Day 1	7	4	0
Day 2	9	6	2
Day 3	12	9	5
Day 4	14	11	7
Day 7	20		
Day 10		20	
Day 16			20
Any day		x	

Fig. 1. The table that Joey had been peeking at.

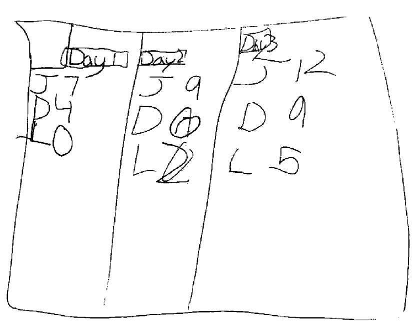

Fig. 2. Joey's table, constructed while peeking at the table in Fig. 1.

2. The study and its context

The work reported in this paper is part of the work we have been doing during the last 3 years with children between 8 and 10 years of age. In our work, we have been exploring some ideas about how to bring out the algebraic character of arithmetic (see Brizuela, Schliemann, & Carraher, 2000; Carraher, Brizuela, & Schliemann, 2000; Carraher, Schliemann, & Brizuela, 1999, 2000; Schliemann, Carraher, & Brizuela, 2000). Most efforts to "algebrafy" arithmetic (Kaput, 1995) have begun with multiplicative reasoning and linear functions. We believe that there is no need to put off the algebrafication of arithmetic until multiplicative structures. Moreover, children's initial intuitions about order, change, and equality do arise within additive situations. During this time, central to our work has been trying to understand how young children work algebraic representations into their thinking. In this paper, we extend these views into an exploration of how children understand, construct, and think about tables in additive situations.

The specific examples that we use in this paper are taken from our work with the children in four second grade classrooms in a public elementary school from a multicultural community in greater Boston. Joey, mentioned above, is one of these second graders. These examples are part of our longitudinal work with students as they move from their second through fourth grade of schooling. During the second grade, we met with the students once a week during a 6-week period. Our curriculum that semester was focused on an exploration of additive structures. Midway through the semester, we interviewed the children in pairs. The purpose of the interviews was two-fold: to learn about the students' understandings of the concepts, and to hold more individualized interactions with the students, as learning opportunities. We interviewed a total of 39 children. During that interview, we presented the students with the following problem — which is the same that Joey, had been presented with. A model table was included at the end of the children's handouts for a different task during the interview, but was not shown to the children during the part of the interview that we will be focusing on here. Fig. 3 shows the problem that was presented to the children.

Day 1:

Jessica, Daniel, and Leslie each has a piggy bank where they keep the money they receive from their grandmother. One day, they counted the money they had and found that Jessica had 7 dollars, Daniel had 4 dollars, and Leslie had none. They then decided not to spend any more money and keep all the money their Grandma would give them in their piggy banks.

Show how much money each one has.

Day 2

On the second day Grandma came to visit and gave 2 dollars to each one of the children. They put the money in their piggy banks.

Show what happened.

Day 3

On the third day Grandma came to visit and gave 3 dollars to each one of the children.

Show what happened.

How much money does each one have now?

Show in a table what happened from day 1 to day 3.

Fig. 3. The problem that was presented to the children.

3. Looking at children's self-designed tables

After going through the problem, we asked each child to show in a table what had happened in the problem, from day 1 through day 3. We did not give the children a model to follow in building their tables, although they had already worked with tables in their classes with us.

3.1. The range of representations

The children's responses spanned the range from the very idiosyncratic, like Jennifer's (see Fig. 4) to the more conventional, like Joseph's (see Fig. 5). Interestingly, Jennifer and Joseph had actually worked together on this problem even though their responses were radically different.

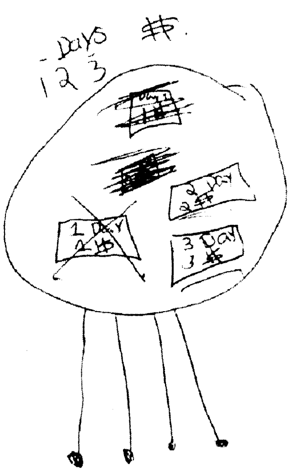

Fig. 4. Jennifer's table.

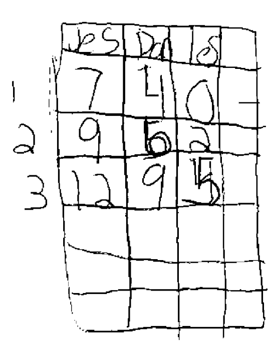

Fig. 5. Joseph's table.

Jennifer explained that she drew a table (literally!), with its four legs. She described the table she made in the following way:

Jennifer: See, I wrote days up here (pointing to the word "days"), and then wrote, there was three days, right (pointing to the numbers)? And that is including money (pointing to the dollar bill). And then I drew squares and it says,... on day two they got two dollars and on day three they got three dollars.

Joseph, on the other hand, explained that he did a "regular table ... it's just got days and names" (maybe feeling slightly intimidated by Jennifer's creation!).

3.2. Following a temporal order

Of the 39 children interviewed, over half of them (22 of the 39) followed a temporal order in their construction of the tables. By this we mean that they placed the names of the characters as headers for the columns and the days of the week as headers for the rows. Thus, going down a column also reflects the passing of time. This is the conventional way to construct tables. Joseph's table is an example of this kind of construction (see Fig. 5). In contrast, an example of a table that does not follow this same kind of order is that of Joey, in Fig. 2.

3.3. Explicit and implicit information

An analysis of the children's tables also indicates what kind of information they made explicit and which remained implicit. In their tables, children were more likely to make the names of the characters in the problem (e.g., Daniel, Jessica, and Leslie) explicit, information that could not be obliterated. Thus, only 3 of the 39 children (8%) left out the characters' names in their tables. An example of a table that left out this information is that of Jessie (see Fig. 6).

In Jessie's table, it is hard to discern who each of the amounts refer to. The fact that there are so few children who left this information implicit may indicate that it is very important for quantities to have some kind of referent or, in this case, "owner" (see Schwartz, 1988, 1996). Along the same lines, 14 of the 39 children (36%) repeat the names of the characters in their

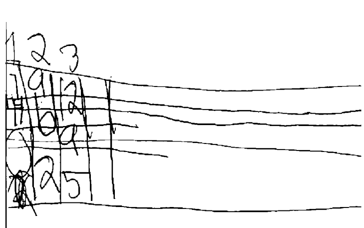

Fig. 6. Jessie's table, which does not show the characters' names.

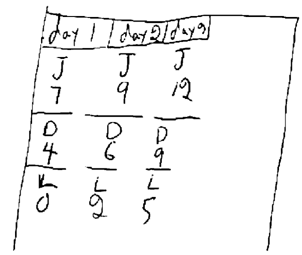

Fig. 7. Raymond's table, which repeats the names of the characters in each cell.

tables. Raymond's table, for example, seems to indicate that he felt compelled to include the names of the characters of the problem — the "owners" of each amount — in each one of the cells in his table (see Fig. 7).

In contrast, the second graders we worked with did not always make explicit the day numbers as relevant. About 13 of the 39 children (33%) did not write the names of the days in their tables (compared to the 3 children who did not write the names of the characters). Briana is an example of a child who did not explicitly register the day numbers in her table (see Fig. 8). This may indicate that in her view, this information did not need to be made explicit.

It seems that the day numbers is information that can be left implicit. Going along the cells (be it down the columns, as would be done conventionally, or across the rows, as was done by some children like Jessie and Raymond) shows the passing of time. Some children may think that therefore this information can be left implicit. Additionally, very few children (2 of the 39 children, or 5%) actually repeat the day numbers in their tables (compared to the 13 children who repeated the characters' names in their tables). Adam, for example, repeated the information about the day numbers in his table (see Fig. 9).

Moreover, when providing a verbal explanation of his representation during the interview, Adam felt compelled to repeat not only the day numbers but also each character's name, each day:

Adam: Day one, Jessica had seven dollars. Day one, Daniel had, uhm, four dollars, and day one Leslie had zero dollars. Day two, Jessica had nine dollars. Day two,

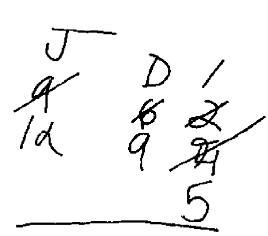

Fig. 8. Briana's table, which does not show the day numbers for the problem.

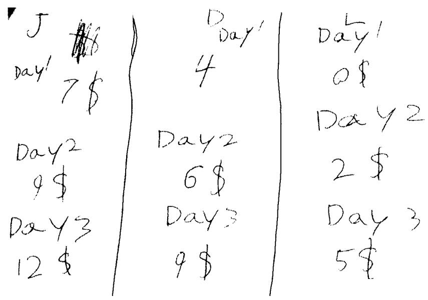

Fig. 9. Adam's table, where he repeats the day numbers in each cell.

> Daniel had six dollars. Day two, Leslie had two dollars. Day three, Jessica had twelve dollars. Day three, uhm, Daniel had three dollars. Day three, Leslie had five dollars.

The choices children make about the kind of information to make explicit and to leave implicit in their tables highlight some of the issues that these children may find relevant in their construction and re-construction of function tables (Ferreiro (1986) and García-Milà, Teberosky, & Martí (2000) have explored previously this issue of explicit and implicit information). The referents for the amounts are important to them, while an indication of the temporal order of the events is not as important. In children's minds, they may find that there are other ways in which tables can indicate temporal order, but not the referents to the amounts included in the table.

3.4. Looking into additive relations

Of the 39 children interviewed, most children (36 of the 39, or 92%) showed in their tables the cumulative amount of money that each child had on each one of the 3 days — that is, how much money each child had and not how much they got. Tables such as Joseph's (Fig. 5), Briana's (Fig. 8), and Adam's (Fig. 9) are examples of tables in which only the cumulative amounts are depicted in the tables. On the other hand, the other three children showed in their tables the amount of money gained each day instead of the total amount of money — that is, how much they got and not how much they had. Jennifer's table (Fig. 4) is an example of this kind of representation. Another example of this kind of representation is that of Mariah, who shows the amounts gained by drawing a circle around them. The circled parts at the bottom of the representation are the amounts the children got from their grandmother on the second and third days (see Fig. 10).

Thus, in thinking about additive relations they were representing in their tables, most of the children focused on total amounts; that is, on how much each child had on each one of the 3 days. A variation of this focus can be seen in representations such as Briana's (Fig. 8) and Jessie's (Fig. 6). In their tables, they crossed out the amount of money the children had

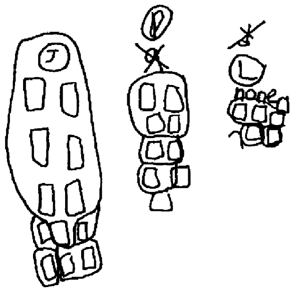

Fig. 10. Mariah's table, showing how much the children got on each day.

on the previous day once they noted how much they had on the following day. Thus, when they moved on to day 2, they crossed out the amount on day 1. In their representations, they seem to need to clarify that the amounts represented in each row should not be added to the amounts in the previous rows, but that each new row makes the previous ones invalid. A few children, however, focused on the differences between the amounts that the children had on each one of the days, or how much the children got on each one of the days. This is the case of Jennifer (Fig. 4) and Mariah (Fig. 10).

4. Concluding remarks

We take seriously and at heart a question posed now more than 10 years ago by Kaput (1991, p. 55): "How do material notations and mental constructions interact to produce new constructions?" While we do not believe that we could answer this question conclusively based on the data here presented, we can indicate that there is a great need to look further and deeper into children's construction of function tables. By developing better understandings both of how their representation of tables evolves and of the understandings about additive relations that are reflected in their representations, we can better serve them as teachers and as curriculum developers. We have begun to unearth some of children's understandings about additive relations reflected in their representations. Still to be further explored is the interaction both of their constructions and of conventional tables on the development of their understandings about additive relations.

Even though their experiences with function tables are counted, the second grade children we worked with have developed sophisticated tabular representations of the data presented in a problem focusing on the progression of money gained over time. They construct function tables that make sense and that reveal the workings of their logic about the problem at hand, and they adequately organize and represent the problem situation. Interestingly, children incorporate into their representations conventional features of function tables. Over half of the children we interviewed followed a temporal order in their tables, in terms of rows (for characters) and columns (for the passing of time). Additionally, information such as showing the characters' names is more frequently made explicit when constructing these tables, which is consistent with Schwartz's (1988, 1996) suggestion regarding the importance of having referents for quantities. Additionally, data tables seem to help uncover children's conceptions and understandings about additive relations.

Acknowledgments

A version of this paper was presented at the 12th ICMI Study Conference held in Melbourne, Australia, in December 2001. Support for this research was provided by NSF Grant no. 9909591 (Bringing Out the Algebraic Character of Arithmetic), awarded to David Carraher (TERC) and Analúcia Schliemann (Tufts University Education Department). Special thanks to Fred O'Meara, Thelma Davis, and the second grade students involved in our project for their help, and to David Carraher and Jim Kaput for constructive comments on earlier versions of this paper.

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