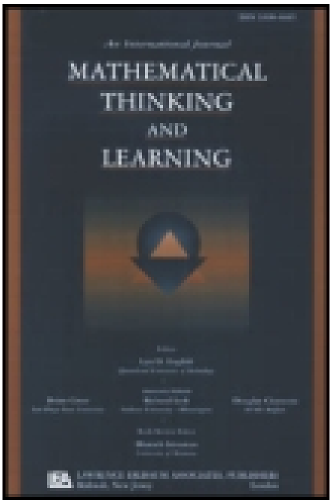

Mathematical Thinking and Learning
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/hmtl20

Children's Use of Variables and Variable Notation to Represent Their Algebraic Ideas

Bárbara M. Brizuela[a], Maria Blanton[b], Katharine Sawrey[a], Ashley Newman-Owens[a] & Angela Murphy Gardiner[b]
[a] Tufts University
[b] TERC
Published online: 30 Jan 2015.

Mathematical Thinking and Learning, 17: 34–63, 2015

ISSN: 1098-6065 print / 1532-7833 online
DOI: 10.1080/10986065.2015.981939

Routledge
Taylor & Francis Group

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Bárbara M. Brizuela
Tufts University

Maria Blanton
TERC

Katharine Sawrey and Ashley Newman-Owens
Tufts University

Angela Murphy Gardiner
TERC

In this article, we analyze a first grade classroom episode and individual interviews with students who participated in that classroom event to provide evidence of the variety of understandings about variable and variable notation held by first grade children approximately six years of age. Our findings illustrate that given the opportunity, children as young as six years of age can use variable notation in meaningful ways to express relationships between co-varying quantities. In this article, we argue that the early introduction of variable notation in children's mathematical experiences can offer them opportunities to develop familiarity and fluency with this convention as groundwork for ultimately powerful means of representing general mathematical relationships.

INTRODUCTION

In this article, we explore the different understandings held by first grade students (approximately six years old) about variable and variable notation[1]— the use of letters to represent variable quantities.[2] Three main issues motivate this exploration. First, variable is a central part of learning algebra. Thus, calls for the development of children's algebraic thinking as early as Kindergarten (e.g., Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000, 2006) bring up the need to understand how children in elementary grades

[1]In this paper, when we say *variable* or *variable notation* we mean both the concept of variable and the notation to represent variables. For ease of reading, throughout the paper, we will use either one of the terms, but mean both.

[2]In this paper, when we say *variable* or *variable quantity* we refer to both varying and fixed unknown quantities (see Blanton, 2008; Blanton, Levi, Crites, & Dougherty, 2011). An example of the former would be the use of x to represent a varying, unknown quantity in the equation $x + 7 = y$. An example of the latter would be the use of x to represent a fixed, unknown quantity in the equation $x + 7 = 10$.

Correspondence should be sent to Bárbara M. Brizuela, 12 Upper Campus Rd., Paige Hall, Department of Education, Tufts University, Medford, MA 02155. E-mail: barbara.brizuela@tufts.edu

understand all aspects of algebraic thinking—including the use of variables. While there is some research that looks at this issue in upper elementary grades, almost no research examines it in the very early elementary grades.

Second, and related to this first point, not only are variables a central part of algebra but variable notation is a powerful tool for expressing generalizations (Carraher & Schliemann, 2007; Dougherty, 2008; Kaput, Blanton, & Moreno, 2008; Pimm, 1995; Tall, 2013). While prior research has found that students in the early elementary grades can use verbalizations to express generalizations (e.g., Radford, 2006, 2011), and while we agree with Radford (e.g., 2011) that "the use of notations (i.e., alphanumeric symbolism) is neither a necessary nor a sufficient condition for thinking algebraically" (p. 310; see also Mason, 1996), we argue that this does not relieve us of the need to examine young students' understandings of variable and variable notation. Indeed, there is promising research elsewhere (e.g., Dougherty, 2010) that suggests that access to variable notation can enable upper elementary students to express generalizations in a way that they may otherwise be unable to do. It stands to reason that this might apply to young children as well.

Third, students in middle grades and beyond have well-documented difficulties with variables and variable notation (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984; Filloy & Rojano, 1989; Kieran, 1985, 1989; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2011; Küchemann, 1981; MacGregor & Stacey, 1997; Steinberg, Sleeman, & Ktorza, 1990; Vergnaud, 1985). In contrast, research with upper elementary grade students (that is, grades 3–5) shows that they can grasp and leverage variable and variable notation in the process of generalizing (e.g., Blanton et al., 2015; Brizuela & Schliemann, 2004; Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, & Schwartz, 2008; Dougherty, 2008; Schliemann, Carraher, & Brizuela, 2007; Sutherland, 1989, 1991). This prior research has shown that variable notation increasingly becomes part of students' repertoire when they are provided opportunities to interact with it on a regular basis. While it is reasonable to assume that an even earlier introduction would give students the time and space necessary to develop robust understandings of variable notation that could preempt the difficulties older students currently face, to date younger students' understandings and uses of variable and variable notation have not been sufficiently explored.

Researchers in the area of early algebra whose work has shown variables to be within the grasp of students in grades earlier than middle school (e.g., Blanton et al., 2015; Cai, Ng, & Moyer, 2011; Carpenter et al., 2003; Carraher et al., 2008; Dougherty, 2008; Russell, Schifter, & Bastable, 2011; Schliemann et al., 2007; Schmittau, 2011; Schoenfeld, 2008) have also described different interpretations that students have of variables and have provided us with significant insights regarding students' capacity to think algebraically. The cited research, carried out with upper elementary school children from different socioeconomic and educational backgrounds, has documented that although students sometimes see letters that represent varying quantities as fixed unknowns or as objects or labels, they can also come to understand letters as representing varying quantities.

As research with upper elementary and adolescent students has done (see, e.g., Carraher et al., 2008; Knuth et al., 2011; Küchemann, 1981; MacGregor & Stacey, 1997; Sutherland, 1989, 1991), we present here analyses of our work with first grade children to illustrate the understandings that young children hold regarding variables. We argue that by looking at young students' understandings we can develop a more complete picture of patterns of early engagement with variable notation upon which children's later understandings could build. Thus, in this article, we address the following research question:

> What are some of the understandings about variable and variable notation held by first grade (6-year-old) students?

Theoretically, our work challenges discussions of "developmental readiness" that have focused on a narrow interpretation of Piaget's work (for a deeper discussion, see Schliemann et al., 2007) and that tend to assume that meaning or conceptual understandings precede the use of symbols or language (for further discussion, see Sutherland, 1989, 1991; Sutherland & Balacheff, 1999). Instead, our work assumes, following mathematics education researchers focused on the mediating role of sociocultural tools (e.g., Kaput, 1991; Kaput et al., 2008), that meanings and symbols co-emerge (see Sfard, 2000), that the relations between (representational) forms and (cognitive) functions shift over time (see Saxe, 1999, 2005; Saxe & Esmonde, 2005), and that symbols can support and mediate the construction of new meanings and conceptual understandings. Further, we agree with Sutherland and Balacheff's (1999) assessment that there is a purposeful role on the part of the teacher when introducing variable and variable notation:

> The teacher has to somehow support students to learn to use the algebraic language and to learn to use this language to develop algebraic ways of solving problems. Our claim is that the algebraic language has to be injected into the situation from the beginning. (p. 9)

Our goal in this article is to illuminate the productive nature of children's understandings and uses of variable and variable notation. To be clear, we are not looking for or even expecting all children to understand variable notation as representing varying quantities—we are interested in the variety of understandings held by children. We explore here how, if given the opportunity, children as young as six years old can use variable and variable notation in meaningful ways—albeit not always as representing varying quantities—to express relationships between co-varying quantities. We explore here whether, in the same way that early exposure to spoken and written words facilitates children's literacy and verbal development, the early introduction of variable notation in children's mathematical experiences can initiate the development of their fluency with this convention as groundwork for powerful ways of representing general mathematical relationships.

BACKGROUND

A central argument for including algebra in the elementary grades is that the traditional "arithmetic-then-algebra" approach does not allow for the slow, deep development of students' mathematical thinking. Instead, six-to-eight years of arithmetic instruction in the elementary grades, followed by an abrupt, brief, and largely superficial introduction to algebra in secondary grades (e.g., Bills, Ainley, & Wilson, 2003; Kaput, 1998) can result in a disconnected, algorithmic understanding of algebra content, rather than an integrated understanding that can bolster the many aspects of algebraic thinking in diverse mathematical domains.

The failure of this traditional approach to algebra is manifested in a widespread marginalization of students who do not perform successfully on algebraic tasks in standardized assessments (Chazan, 2000; Kaput, 2008; Moses & Cobb, 2001). The victims of this mathematical marginalization include students from all demographics, and particularly from under-represented minority groups and low-income households (Diversity in Mathematics Education Center for Learning and

Teaching, 2007; Moses & Cobb, 2001). In effect, students are often first marginalized from algebra, later marginalized from calculus, and ultimately marginalized from a host of higher education opportunities (Schoenfeld, 1995).

Very frequently, students' failures in formal algebra are related to their difficulties with understanding variables and variable notation (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984). Students in these traditional algebra experiences (i.e., not early algebra) have often been judged on their ability to manipulate symbols, usually very soon after a superficial introduction to variable notation. It is paramount that early algebra avoid the procedural tricks that have been deeply problematic for these older students. In other words, (early) algebra should allow children to experience algebra as a meaning-making activity instead of as a series of algorithmic steps or procedures (Schoenfeld, 2008). However, because meanings cannot be considered separately from the ways in which we represent them (e.g., Sfard, 2000; Vygotsky, 1978), we argue that variable notation needs to be integral to these early algebra experiences as a tool through which children learn to express and reflect on meanings.

Recent research has also shown that when late elementary school students (grades 3–5) have had early algebra experiences that include regular use of variables, they outperform comparison students on assessment items that include, among others, variables and algebraic expressions (Blanton et al., 2015; Brizuela, Martinez, & Cayton-Hodges, 2013). In addition, when followed in subsequent studies, students have been found to outperform comparison students two and three years after their early algebra instruction has ended (Schliemann, Carraher, & Brizuela, 2012). This evidence, along with the theoretical perspective that meanings and the ways in which we represent them go hand in hand, leads us to argue that variable notation needs to be part of students' early algebra experiences.

There are those who advocate that prior to middle or high school, students should use natural language—in lieu of variable notation—to represent their algebraic ideas. From this perspective, delaying the introduction of variable notation is justified based on the view that such notation is developmentally inappropriate or likely to engage students in meaningless symbol manipulation (e.g., Linchevsky, 2001). As research suggests, however (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984), even when the introduction of variables is postponed to later ages, there is the potential for variable notation to be learned as a series of algorithmic tricks rather than as a productive representation of and for thinking. This is unfortunate, and for us, underscores that the issue is less about the age of the student and more about the role of meaning within learning. As such, there is a need for studies, such as that reported here, that document children's nascent understandings and can point to alternative models for developing deep understandings of variable and variable notation.

In summary, despite all we have learned in recent decades regarding the need and importance of an early introduction to algebra, its representations, and its practices, young children's understandings of variable and variable notation have remained largely unexplored. This article explores children's understandings and uses of variables as they examine functional relationships. It complements research on upper elementary school children's use of variable notation in this particular mathematical domain (e.g., Blanton et al., 2015; Brizuela & Schliemann, 2004; Carraher et al., 2008; Cooper & Warren, 2011; Moss, Beatty, Shillolo, & Barkin 2008; Moss & McNab, 2011; Schliemann et al., 2007). Moreover, it complements prior research on children's use of variable notation as they generalize arithmetic relationships (Carpenter et al., 2003) and as they investigate and represent quantitative relationships (Dougherty, 2008, 2010). As such, it has the potential to give us a more comprehensive picture of how, from very young ages, children understand variables. To explore our claims, we analyze a first grade classroom episode in which variables and variable notation were used and discussed, as well as individual interviews with students who participated in that classroom event.

METHODS

Participants

This article draws data from a larger study carried out in a public school in the Northeast region of the United States. At the time of the study, 1% of the school population was White, with a majority either African American (47.3%) or Latino (46.8%). Additionally, at the time of the study, 77.8% of students in the school were on free or reduced lunch. The year we carried out our study, 52% of third grade students were considered "warning/failing" on a state mandated standards-based test in mathematics. Another 40% were considered "needing improvement." We worked with one classroom of 18 Kindergarten, one classroom of 22 first grade, and one classroom of 21 second grade students. The data for this article are drawn entirely from the first grade classroom, as a way to reduce the data (e.g., Chi, 1997) and engage in a more in-depth analysis of a smaller data set.

Our Teaching Experiment

We used design-based research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, 1992; Kelly, 2003) to develop and carry out a classroom teaching experiment at each of grades Kindergarten–second grade (K–2). The experiment involved eight weeks of classroom instruction organized into an instructional sequence that consisted of two 4-week cycles (with two 40-minute lessons taught per week), along with selected student interviews before, between, and after the two cycles of instruction. The tasks designed for the teaching experiment were based on field-tested tasks developed in previous early algebra research by the project researchers. The same lessons were taught in all classrooms across grades K–2.

Six of the 22 first grade students were interviewed before, between, and after the two cycles of instruction. The purpose of the individual interviews was to gain an in-depth perspective into the individual thinking of a subset of students, as opposed to their thinking in a group setting. Selection for children to interview at each moment in the study was based on a combination of the classroom teacher's nomination of the top third of academically performing students—as indicated by their numeracy and verbal skills—and the research team's selection of students who were verbally expressive in class, participated actively, and readily volunteered their thinking. Thus, the children we ended up interviewing were not necessarily considered the top-performing students by their teachers. We asked initially for the classroom teacher's nomination of the top third of students in each class to ensure that the students we interviewed would be able to both engage in the tasks and provide us with clear articulations of their reasoning.

Six children participated in the pre-interview, focused on an indentity function, $y = x$. Subsequently, all children participated in the first four-week cycle of instruction, which consisted of eight lessons. The first two lessons introduced the idea of co-varying relationships by reviewing geometric pattern sequences. The remaining six lessons in this cycle focused on functions with multiplicative relationships of the form $y = mx$.[3] Upon completion of the first cycle, six students participated in a mid-interview that focused on a function of the type $y = x + b$.[4] We then implemented the second four-week cycle, which also consisted of eight lessons. All lessons in this cycle were based on functional relationships of the form $y = x + b$. At the conclusion of the second cycle, six students participated in a final interview focused on $y = mx$ and $y = mx + b$ functions. Figure 1 summarizes the structure of the teaching experiment.

Each of the 16 lessons implemented in our classroom teaching experiment (see Figure 1) focused on a problem scenario (see, e.g., Appendix A and the scenario presented and described in the classroom episode below). Once the scenario was presented to the students, the research team member who was acting as teacher/facilitator engaged the students in a discussion about what they understood about it, possible pairs of values that could instantiate the underlying co-varying relationship, and a general rule that could be used to represent the underlying relationship. During this discussion, the facilitator often used a flip chart or white board to keep track of students' ideas and a function table to keep track of possible pairs of values suggested by the students. Additionally, the facilitator recorded equations generated by students to represent a relationship between a pair of values (e.g., "3 + 3 = 6").

The interviews we carried out were semiclinical in nature. In them, we also presented students with problem scenarios akin to those presented in class. We followed a protocol of questions that reflected the following steps: (1) explore the student's understandings of the scenario; (2) ask for his or her own representations of the problem; (3) ask the student to organize the information in some way; (4) ask the student to organize the information in a function table; (5) ask the student to describe a general rule that could represent the function; (6) ask the student to express this rule using variable notation; and (7) ask the student to describe the meaning of the rule within the problem scenario. The interviewer supplemented the interview protocol by asking follow-up questions to explore each child's thinking about the problem in more detail.

Lesson	Pre-interview	First 4-week cycle (8 classroom lessons)	Mid-interview	Second 4-week cycle (8 classroom lessons)	Post-interview
Function type	$y = x$	$y = mx$	$y = x + b$	$y = x + b$	$y = mx$ $y = mx + b$

FIGURE 1 Structure of the classroom teaching experiment.

[3]Because of the age of the participants, we expected them to express these relationships additively rather than multiplicatively. For example, the relationship $y = 2x$ might be represented as $y = x + x$.

[4]We chose functions of the type $y = x + b$ instead of $y = b + x$ for two reasons. First, they more closely model the mathematics that children are accustomed to engage in. For instance, for the example provided in this paper, if we have a variable height, and a constant hat height, it is more likely that children would look to symbolically express the following:

"what is someone's total height (y) if they (x) put on a hat that is one foot tall (b)," and not this:

"what is someone's total height (y) if we take a one-foot-tall hat (b) and put it on them (x)."

Second, in our own prior research (e.g., Blanton, 2008; Blanton et al., 2011; Schliemann et al., 2007) with upper elementary school children, we had found that students had no difficulties making sense of functions of the type $y = x + b$.

Data Sources and Selection

All classroom lessons and interviews were videotaped and all written student work was collected. All interviews (pre-, mid-, and post-) were transcribed verbatim. We first reviewed the videotapes of lessons in our teaching experiment. This first review of videotapes led to a progressive selection of the data set for this article. In this selection process, we adopted an approach similar to that described by Nemirovsky, Kelton, and Rhodehamel (2013), whose goal was "not representativeness but rather the enrichment of the reader's own perception" (p. 385). The criterion we used to select the data for this analysis was that they exhibit a wide range of understandings about variable and variable notation among an academically diverse set of students. Our goal was not to showcase only the most sophisticated thinking in young students, but the range of understandings, as had been done previously for upper elementary (e.g., Carraher et al., 2008; Sutherland, 1989, 1991) and adolescent students (e.g., Knuth et al., 2011; Küchemann, 1981; MacGregor & Stacey, 1997).

To further facilitate data reduction (Chi, 1997), we chose to focus on data from first grade. We selected first grade (over second grade) because we sought data from students as close as possible to the start of formal schooling, but still relatively articulate (i.e., not Kindergarten students, whose limited language skills sometimes made it challenging for them to clearly express their thinking). Further, by focusing on a single classroom, we were able to investigate the range of understandings among a group of students who had had a shared classroom experience.

Once the decision was made to focus on first grade participants, we further reduced our data corpus to the first half of the second four-week cycle of our classroom teaching experiment that concentrated on functions of the type $y = x + \mathrm{b}$ (see Figure 1). We were interested in children's understandings as these were beginning to develop, more so than in either their absolute first attempts to engage with variables and variable notation, or the most sophisticated understandings they achieved by the end of the study. By this choice, we acknowledge that we are not trying to showcase the most sophisticated understandings that children in our study achieved. During this phase (i.e., weeks five and six) of our teaching experiment, children were beginning to use variable notation during classroom discussions and to represent algebraic relationships. Elsewhere (e.g., Blanton, Brizuela, Sawrey, Newman-Owens, & Gardiner, in press), we identify a trajectory in children's thinking that documents that the level of sophistication they exhibited extends beyond what we discuss here. For example, in Brizuela, Blanton, Gardiner, Newman-Owens, and Sawrey (in press), we report on a first grader who by the end of our teaching experiment understands letters as representing varying quantities.

Subsequently, we selected Lesson 12, which occurred during week six of the eight-week sequence and focused on a $y = x + 3$ function. Of the eight lessons in the second four-week cycle of our teaching experiment, children's participation in Lesson 12 illustrates the widest range of understandings we were able to document. In this lesson, the last 5–10 minutes were spent discussing how to use variable notation to represent the situation. This is the episode we focus on in this article.

Once this specific episode had been chosen as a focus for analysis, we also looked for interviews of the four students who participated in the discussion that took place in class—Zyla, Nyah, Leo, and Tia.[5] Of the interviews, we selected the mid-interview because it focused on a function

[5] All children's names used are pseudonyms.

that was structurally the same as the one we focused on in Lesson 12: $y = x + 1$. This allowed us to capture a more detailed look at first graders' understandings about functions with the same structure at different moments in time. In addition, the mid-interview helps to paint a portrait of how children were envisioning variables and variable notation before Lesson 12 and before their introduction to functions of the form $y = x + b$ in a classroom setting.

The mid-interviews with Zyla, Nyah, Leo, and Tia were four of the six mid-interviews we carried out with first graders. These four students were highly verbally expressive[6] during the Lesson 12 discussion. Of these four students, only two (Tia and Leo) had been nominated by their teacher as within the top third of the class. Zyla and Nyah were considered by their teacher to be in the middle or bottom third of the class academically. Our own observation was that Zyla, Nyah, Leo, and Tia did not always exhibit the most sophisticated thinking during our lessons, but they were always eager to participate and share their thoughts in class.

Analysis

Once the data were selected, we began a line-by-line review of the transcripts (with periodic review of the video clips associated with the transcripts) and tagged any moments in the transcripts when references were made to variable notation. We tagged these moments using the constant comparative method (Glaser & Strauss, 1967) to identify key features in how students discussed and used variables and variable notation. We used this approach to describe the range of understandings displayed by these first grade children. Once these understandings had been listed, coded, and identified in the transcripts, we began to create a detailed narrative account (or "thick description," see Geertz, 1973) that could help explain the contexts in which these understandings were manifested.

In the mid-interview and lesson episodes we analyze next, we present extended conversations—as opposed to shorter pieces interjected with commentary and analysis—to provide the reader with a close sense of the moment-to-moment events and a more transparent perspective on our interpretation of the data.

RESULTS

From the outset of our teaching experiment, we brought up variable notation in every lesson once children had worked with specific pairs of values that could instantiate the underlying relationship and had talked about their understandings of the general relationship between the quantities. We were not looking for canonical or normative articulations of the general rule, but children's impressions of what they were noticing about the variation between the two quantities. We would then ask the children to consider how they might articulate the general rule using variable notation. Therefore, by the time the mid-interviews took place, there was a routine practice of introducing variable notation in all our lessons.

[6]Certainly, the thinking of students who are not visible (verbal) participants in a classroom discussion is an interesting point of study. While it is not our intent to minimize this aspect of learning, it is beyond the scope of our paper.

Mid-Interviews

The mid-interview task, “Height with a Hat,” involved a scenario in which we considered people’s heights if they were wearing a hat that was one foot tall (so, a function that could be represented as $y = x + 1$, where x represents a person’s height in feet and y represents their height when wearing the one-foot hat).

Zyla’s Mid-Interview

When asked to explain what one needs to do to find a person’s height with a hat given the person’s height, Zyla explained that you need to: “Add . . . a plus sign. . . . The—plus . . . plus one.” The interviewer then continued by trying to get Zyla to consider how “any height” could be represented:

Z1: Interviewer: What if I said that it’s “*Z* feet tall” [writing *Z* in the left hand column in a two column table] . . . what should I do right there [pointing to the right hand column, right next to *Z*]?
Z2: Zyla: I would put “one *Z* tall.”
Z3: Interviewer: Can you show that?
Z4: Zyla: [Writing what appears in Figure 2. She first writes the “*Z*,” then says “w—oh!” and goes back and writes the word “one” in front, and then “tall.” She adds the “hat” later when the interviewer asks her to clarify what the word “one” represents—“The one is the hat . . . One hat [writes “hat”].] One . . . one *Z* . . . ! [Laughs.]

When asked what she meant by the representation “one *Z* tall,” Zyla explained that the “one” in Figure 2 “is for because this [gesturing the height of the hat] is one foot.” She explained that she had included “*Z*” in her production “’Cause you put *Z* right here,” and moved the interviewer’s finger as she said this to point to the *Z* the interviewer had written in the left hand column of the table. She seemed to indicate that her own response had to incorporate a *Z*, given that the interviewer had introduced one. She then explained that it meant “tall” and finally “how . . . *Z* people are . . . so you put . . . so how tall *Z* people are [pointing to “how tall people are,” her own label for the column in the table in which the interviewer had included *Z*].” In this excerpt, Zyla seemed to be using *Z* to refer to the number of people (i.e., “how tall *Z* people are”). Zyla was therefore understanding the variable notation *Z* as representing a quantity. We make this interpretation of Zyla’s “how tall *Z* people are” because a use of *Z* to represent a number of people would be consistent with prior lessons in which the variable had been verbally described as “the number of (days, dogs, coins).” An alternate possibility is that when Zyla said “how tall *Z* people are” she meant *Z* people to refer not to the number of people but to some particular type of people who can be called “*Z* people.” However, given the references to variables in previous

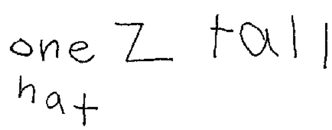

FIGURE 2 Zyla’s production of “one *Z* tall.”

lessons, we find it both plausible and likelier that Zyla meant "number of people" when she said "how tall *Z* people are."

Later in the interview, Zyla provided us with several examples of the fact that she was unwilling to accept indeterminate quantities (Radford, 2011). Instead, she looked to instantiate quantities. That is, each time the interviewer attempted to get her to consider an unknown and possibly varying quantity, Zyla's inclination was to provide a specific, fixed quantity instead, as in her suggestion below that we just measure the person:

Z5: Interviewer: What if I don't know how tall someone is? What do I have to do to tell how tall they are—
Z6: Zyla: You would tell them to lay down—
Z7: Interviewer: Uh-huh . . .
Z8: Zyla: —and then, actually, you could put a paper down [and measure them]—

Later in the interview, Zyla continued to insist that measurement could provide us with a specific, fixed quantity:

Z9: Interviewer: But what if I want to know how tall I'm gonna be with the hat on?
Z10: Zyla: Well, you can just tell them to stand up, but then I measure them with blocks 'cause they—
Z11: Zyla: Okay, mm-hm.
Z12: Zyla: —'cause they stand up [gesturing with hands].
Z13: Interviewer: And then? What do they do to find out how tall they're gonna be with the hat on?
Z14: Zyla: Huh?
Z15: Interviewer: So, what if I wrote here [in the left hand column of a table she was constructing]: "any number"—
Z16: Zyla: Any number?
Z17: Interviewer: Any number.
Z18: Zyla: So, I would put 12!

In this exchange, Zyla suggested either measuring the person (either by having them lie down on the floor and tracing their outline and then measuring that or by having them stand up and measuring them using blocks) or assigning their height, presented by the interviewer as "any number," a specific value: 12.

However, Zyla's attitude toward indeterminate quantities shifted once variable notation was introduced. She never had the inclination of instantiating values when using variable notation. That is, she was willing to leave the quantity indeterminate when variable notation was used, even though she was inclined to instantiate the measure of a quantity in the absence of variable notation when natural language was used, as shown. For instance, a little later, the interviewer made a suggestion:

Z19: Interviewer: So what if someone said to you that if someone is *Z* [writing] tall, then with the hat, it's *Z* plus one [writing $Z + 1$]. What would you think?
Z20: Zyla: It would be *Z* plus one [pointing at the table where the interviewer has written $Z + 1$].
Z21: Interviewer: Does that make sense or is that totally silly?
Z22: Zyla: It's just lame.
Z23: Interviewer: Why?

Z24: Zyla: Becau— it's lame [shrugs] because— it's lame because . . . because *Z*, 'cause [it] shouldn't be with number, because numbers are different than, um letters. . . .

Here, Zyla did not reject the use of a letter to represent a quantity and did not reject leaving the quantity indeterminate. Instead, she questioned the combination of letters and numerals in the same mathematical expression: "because *Z*, 'cause [it] shouldn't be with number" (see Z24). This may help to explain why Zyla had earlier expressed the "height with a hat" as "one hat *Z* tall" (see Figure 2), including a number written in words (one) instead of a numeral 1.[7]

During one moment in this interview, Zyla also provided evidence that she used variable notation as a label or object. When the interviewer suggested the notation $Y + 1$, Zyla responded, "So the person *Y*, le— let's just pretend their name is *Y*—then the person [*sic*] name is *Y*. So, the— the hat, that hat is one foot tall, so it does make sense because the person name [*sic*] is *Y*. . . ." This, however, was the only moment in the interview when Zyla appeared to assign this meaning to variable notation.

In summary, in her mid-interview Zyla provided evidence of understanding variable notation both as standing for an indeterminate quantity and as a label or object (e.g., Knuth et al., 2011; Küchemann, 1981; MacGregor & Stacey, 1997; Radford, 2011). The fact that Zyla was willing to work with indeterminate quantities represented through variable notation is significant. According to Radford (2011),

> What characterizes thinking as algebraic is that it deals with *indeterminate* quantities conceived of in analytic ways. In other words, you consider the indeterminate quantities (e.g., unknowns or variables) as if they were known and carry out calculations with them as if they were known numbers. (p. 310; italics in original)

Zyla did not reject the use of a letter in a mathematical expression per se, although she did question the combination of letters and numerals in the same expression (see lines Z22–Z24). She understood the use of letters as meaningful within a limited set of situations. These understandings coexisted, and Zyla drew from different ones depending on the specific moment in the interview, task at hand, and question from the interviewer. The lack of stability, shifts, and multiple simultaneous interpretations are consistent with descriptions of knowledge as "pieces" (see diSessa, 1993; Wagner, 2006) and do not take away from the importance of acknowledging the constructive role that Zyla's ideas and understandings had, and that we seek to recognize.

Nyah's Mid-Interview

When Nyah was asked for the general rule for the problem (i.e., "What's the rule for every single number?"), she responded, "The number above it," indicating the next counting number on the number line that was hanging in her classroom. The interviewer then introduced variable notation into the interview, and the following exchange ensued:

[7] Another possible interpretation for Zyla's rejection of $Z + 1$ is that for her, additions need to be represented in their executed form (such as $3 + 1 = 4$). If the result of the operation needs to be made explicit, then just stating $Z + 1$ makes no sense: "it's just lame." However, in classroom lessons we had routinely used unexecuted expressions as a way to examine problems, so it is unlikely that at this point in our teaching experiment Zyla was questioning the unsolved nature of this expression.

N1: Interviewer: Right, and we haven't measured it, so we don't know, and I want to show it for anyone, for every single person. So I'm gonna use *Z* for the height. If I say that the height is *Z*, what would be the height with the hat on?
N2: Nyah: It would be *Z*.
N3: Interviewer: Mm-hm.
N4: Nyah: The reason why it would be *Z* is because after *Z*, there's nothing above it.
N5: Interviewer: Ohhh.
N6: Nyah: So there has— it has to be *Z* again!
N7: Interviewer: Okay, can you put that right there?
N8: Nyah: Zeeee. [Writes *Z* in right hand column in her table.]

Nyah's reference to "there's nothing above it" is connected to her rule regarding the "number above it." Similar to her looking for the next counting number when using specific values, in this case it seems that she was looking for "the next letter" in the alphabet, and since there is no letter after *Z*, the height with a hat of someone whose height is *Z* feet tall has to be *Z* as well. Nyah was willing to leave the quantity that Z represents indeterminate. However, when she constructed an expression that included Z, the outcome of the operation had to be expressed based on the ordinal relationship between letters in the alphabet. This interpretation was confirmed in the exchange that followed immediately:

N9: Interviewer: Okay. [Laughs.] So let me do it with another different letter. How 'bout if I used. . . . I'm gonna put the same thing: "Number of—"
N10: Nyah: Make it tough.
N11: Interviewer: Oh, make it tough? Okay, I'm gonna make it really tough. . . . Okay, so what if I said that the height was *Y*? What would be the height with the top hat?
N12: Nyah: *Z*.
N13: Interviewer: Can you put that right there?
N14: Nyah: [Writes *Z* in right hand column.]
N15: Interviewer: How come?
N16: Nyah: 'Cause after *Y* comes *Z*.

While Nyah left the independent variable represented by *Y* indeterminate, she was using the order of letters in the alphabet, as well as the rule she established at the beginning of the interview, to determine the notation for the dependent variable as "one more" than *Y*. As the conversation continued, she kept employing this understanding. For instance, when the interviewer changed the problem to involve a two-foot-tall hat, and presented the case of a person who was *A* feet tall, Nyah stated that the total height with the hat would be *C*, and that if the person's height were *B*, their total height with the hat would be *D* (see Figure 3).

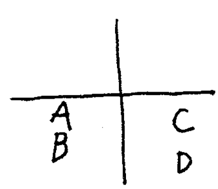

FIGURE 3 Nyah's pairs of values when the hat is two feet tall.

In this interview, Nyah provided us with evidence that she was willing to leave the quantity represented by variable notation indeterminate. This indicates that Nyah was not assigning a specific value when variable notation was used, although doing so is a frequent approach used by elementary to middle school students (see, e.g., Carraher et al., 2008; Knuth et al., 2011; Küchemann, 1981). When constructing algebraic expressions, she was using the order of letters in the alphabet to make decisions about how to express the outcome of an operation, even though the independent variable's value was left indeterminate. This is illustrated in Nyah's work shown in Figure 3. MacGregor and Stacey (1997) reported that 11-to-15-year-old students assign a specific value to letters depending on their position in the alphabet—$A = 1$, $B = 2$, and so on. In Nyah's case, she left the value of the independent variable indeterminate, but her choice of letter for the dependent variable was determined by the "distance" between letters in the alphabet, so $A + 2 = C$ and $B + 2 = D$. While it might be easy to minimize the significance of Nyah's "alphabet" approach from a canonical lens of "correctness," we argue that it is a significant sense-making response. Nyah was not rejecting the use of letters, but was instead interpreting letters based on a point of reference—the alphabet—with which she had had the most experience at that moment. Nyah attached to letters used in a mathematical context a meaning that she was already familiar with and that derived from her prior experiences.

Tia's Mid-Interview

As with the other mid-interviews, Tia and the interviewer explored several specific examples, and captured the numbers in a table (see Figure 4). Although the structure Tia drew looks like a standard, two-column table with headings "Number of People" and "Number of Hats," she captured three categories of information: the heights used in the specific cases (left-most column), the height of the hat (seen as a column of 1's), and equations showing the calculation of a person's height with the hat (for example, $3 + 1 = 4$).

When the interviewer suggested the case of a person with height *a*, Tia added the expression $a + a = ?$ to her table. Looking at Tia's table without any reference to the conversation, we see that Tia created an equation that mimics the additive form of her other equations. One might assume that she would write $a + 1 = ?$ if she were truly trying to match the form of her other equations. However, as we saw with Zyla's and Nyah's interviews, some students—although certainly not all students in our teaching experiment—tended to avoid combining numbers and letters in the same equations. This might help explain why Tia avoided the expression $a + 1$, although she did not state this explicitly. Tia could also have been using *a* to represent 1, as the first letter in the alphabet, although again she did not state this explicitly. The interview does not provide us with enough evidence regarding these two possible interpretations. As seen in the following exchange, however, Tia indicated that she was thinking of variable notation as representing "any number" and an indeterminate quantity:

T1: Interviewer: What if I did something Miss Bárbara's [the teacher/researcher for the classroom teaching experiment] done, and I said that the person was *a* [implicitly referencing *a* feet tall; writes *a* in the left hand column in Figure 4]. What goes on the other side?
T2: Tia: So, the *a* hat. *a* plus *a* equals [writes $a + a = ?$ on the right hand column of Figure 4].
T3: Interviewer: [Pause.] That's awesome. Now, how did you think to do that? Can you explain?
T4: Tia: Because it could be any number.

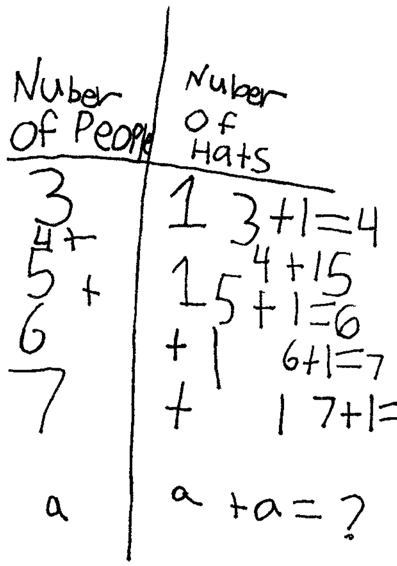

FIGURE 4 Tia's table from the mid-interview.

T5: Interviewer: Okay.
T6: Tia: So we don't know.

Both the interviewer and Tia vaguely referenced the case in which the letter was used. The interviewer did not qualify whether the person was named *a* or had a height *a* (line T1). However, she did write the letter *a* in the left column of Tia's table, implying she meant *a* to represent the person's height (even though the label Tia had given that column was "Number of People," perhaps recalling, as described, previous lessons in which we had focused on "number of (days, dogs, coins)"). Tia responded, "So, the *a* hat," which could mean a specific hat identified by the label *a*, or a hat *a* tall. In her statement, Tia indicated that the letters stand for "any number" (see T4). She also seemed to indicate that *a* is a quality of the hat: "the *a* hat" (see T2). Tia's final assertion in this episode, in response to the interviewer's question regarding whether she meant that *a* could be any number, was that "Yeah, except we don't speak in letters." She seemed to indicate that letters are not usually used in math, stating this with nonchalance. She may have been expressing that using letters in a mathematical context was a novel situation for her, even though she did not seem flummoxed by the experience. In her interview and the equation she produced, Tia provided us with evidence that she accepted the use of letters in mathematical expressions. Tia also left the value of the independent variable indeterminate and assumed that the letter could represent any number.

Leo's Mid-Interview

When Leo, another student in the same first grade class, had successfully considered several specific cases in the "Height with a Hat" scenario and was asked for the general rule, he responded as follows:

L1: Interviewer: Okay, so if we put the hat on anybody, do they get taller?
L2: Leo: Yes.
L3: Interviewer: And how much taller do people get when we put the hat on?
L4: Leo: Um, they get the number of— next to the number they have without the hat.

Here, Leo provided a general statement that could apply to any input value: “they get the number . . . next to the number they have without the hat.” As the interviewer questioned Leo as to how one might show an unknown height or determine a person’s height with a hat without knowing the person’s actual height, Leo shifted from thinking of a general procedure, as he did in L4, to suggesting—like Zyla—a procedure for measuring the person’s height and proposing specific values:

L5: Interviewer: How do you know how to figure out how tall somebody is when they put the hat on?
L6: Leo: Um, because I— I— I tell them to stand up so I can measure them, ‘cause—
L7: Interviewer: You measure them?
L8: Leo: —‘cause, like I’m three feet, I’m three tall. They— they can be like— they can be, I’m three and they’re like f— two, there could be like—
L9: Interviewer: A little shorter?
L10: Leo: It— they could— yeah.
L11: Interviewer: Uh-huh. And if they put the hat on, if they were two and they put the hat on, how big would they be?
L12: Leo: Three like me.
L13: Interviewer: So how do you know to do that every time? What’s the rule?
L14: Leo: Measure.
L15: Interviewer: Measure? But how come you know to add one every time [referencing Leo’s statement in L4]?
L16: Leo: ‘Cause our hat is just one feet.

Although Leo’s explanations suggest that he may have been comfortable leaving quantities indeterminate (“they can be like . . . [two]” [L8]), he supplied specific possible values rather than leave the quantities indeterminate with no values suggested. It is possible that the interviewer’s question “How do you know how to figure out” suggested to Leo—although this was not the interviewer’s intention—that he needed to find a numerical value and measure. Measuring, in fact, is a perfectly plausible response. If you don’t know how tall someone is, just measure them! (Carraher, Schliemann, & Brizuela, 2005 reported on a third grade student giving this same, very reasonable response to a similar question.) However, when the interviewer supplied a letter and suggested that it could be any number, Leo did not supply any value for it:

L17: Interviewer: What if I said that the Measure Number[8] was *Y*, and that could stand for any number?
L18: Leo: *Z*.
L19: Interviewer: It [the height with the hat] would be *Z*? Okay.

Leo did not take the interviewer’s suggestion that *Y* could stand for any number as an invitation to supply a value. Instead, he answered a question the interviewer had in fact not even explicitly

[8]This phrase is how Leo and the interviewer had come to refer to hatless heights.

asked—how tall this *Y*-foot person would be with the hat—and moreover, answered it using a letter. His responses were similar to Zyla's: when variable notation is used, quantities are left indeterminate; when variable notation is not used, the inclination is to measure and supply specific values, although, as noted, Leo might have been influenced by the way in which the interviewer posed the question to him ("How do you know how to figure out . . ." [L5]).

The letter Leo chose was in keeping with the rule he had expressed for numbers. Recall that he explained that people get "the number . . . next to the number they have without the hat:" in this case, he chose the letter next to the letter "they have without the hat." Like Nyah, he was using the order of letters in the alphabet to make decisions about how to express the outcome of the operation, even though the independent variable's value was left indeterminate. We claim that both Nyah's and Leo's responses make a lot of sense given the resources they had available to them and the prior experiences they had had with letters. Leo seemed not to be assigning numerical values to these letters but rather to be letting them occupy the field of things that can represent height. That is, just as three feet (for instance) could be the measure of a height, it seems that for him *Y* feet could also represent the measure of a height. Moreover, all things in this field (i.e., letters and numbers) abide by the same rules. Leo's response, "*Z*," reflected the *Y*-foot person's height with a hat, and he recorded it as such in his table (see Figure 5).

Like Nyah, whose description of the rule was also phrased in terms of nearness ("the number above it"), Leo incremented *Y* by one alphabetic step to produce *Z*. Furthermore, Leo went on to show that his alphabetic incrementing was not confined to cases in which it was only necessary to increment to "the [letter] next to the [letter] they have without the hat." The interviewer later asked him to consider a three-foot hat and a person of height *B*. Leo's response was that the height with the hat would be *E* "'Cause three I counted inside my head" (see Figure 6, left hand side). Now that the dependent value was no longer "next to" the independent value, Leo described "counting three" within the alphabet to get from *B* to *E*. Similarly, for a person with height *L* who was wearing a three-foot hat, Leo offered that the total height would be *O*. These responses are

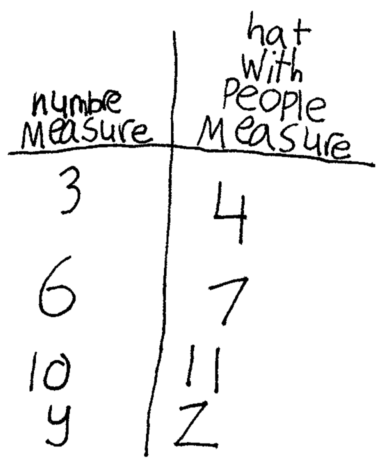

FIGURE 5 Leo's table for the "Height with a Hat" problem (the labels read "measure number" and "measure people with hat").

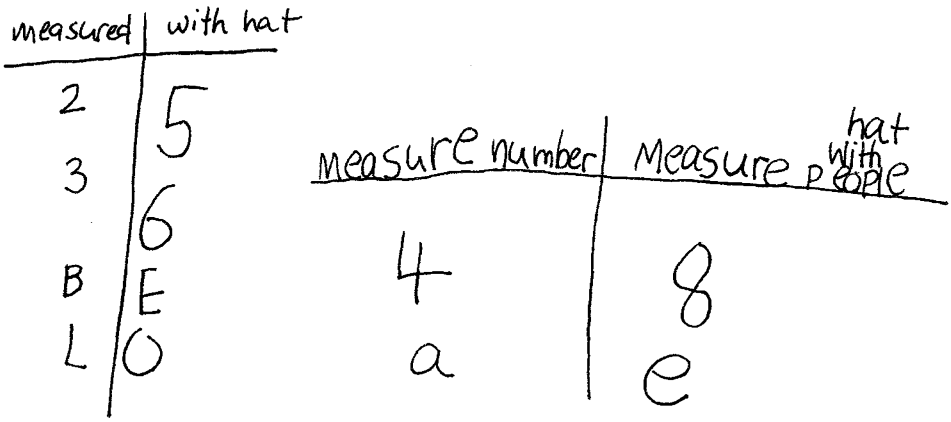

FIGURE 6 Leo's work with a 3-foot hat (left) and a 4-foot hat (right).

further evidence that Leo was systematically, uniformly applying the very reasonable strategy of incrementing each letter the interviewer presented by the number of alphabetic steps equal to the height of the hat. He went on to do the same with a four-foot hat, incrementing *a* to *e* (see Figure 6, right) and *L* to *P*. Again, he did not seem to believe that the letters had or needed specific numerical values.

The interviewer then wrote an expression employing both a letter and a number:

L20: Interviewer: Okay. So this is a three-foot hat? . . . What if try something. If I did [writes] $C + 3$, what do you think about that?
L21: Leo: Mm-hm.
L22: Interviewer: What do you think?
L23: Leo: [Thinks.] *F*.
L24: Interviewer: *F*? Okay, can you write it?
L25: Leo: Equals *F* [writing].

Leo displayed no objection to a mathematical expression that included both a letter and a number; he accepted this expression and built on it to form an equation whose sum was given by a letter (i.e., $C + 3 = F$). Like Nyah, he used a strategy of "counting" forward in the alphabet to make decisions about the outcome of a transformation given a particular input letter, and applied this strategy not just to increment by one, but seemingly to increment by any hat height (in our evidence, to increment by three and four as well).

Summary of Mid-Interviews

Table 1 summarizes the understandings expressed by the four first-grade students whose mid-interviews were just presented. As shown, there is considerable overlap in the understandings expressed. Most remarkable is the fact that all students were willing to leave quantities indeterminate when variable notation was used. Zyla's and Leo's cases are especially illustrative in the shift they displayed when using, versus not using, variable notation: when variable notation was not used, they felt compelled to measure and provide specific values

TABLE 1
Understandings About Variable Notation Expressed by First Grade Students in the Mid-interviews

Understandings about variable notation	*Zyla*	*Nyah*	*Tia*	*Leo*
Variable notation as a label or object	X			
Variable notation as representing an indeterminate quantity	X	X	X	X
Uses the order of letters in the alphabet to make decisions about how to express the outcome of the operation, even though the independent variable's value is left indeterminate		X		X
Avoids combining letters and numbers in a single equation	X			

for the quantity. When variable notation was used, they were willing to leave the quantity indeterminate.

Also of note in the understandings children showed in the mid-interviews is that, in keeping with descriptions of knowledge as "in pieces" (see diSessa, 1993; Wagner, 2006), the understandings are not necessarily stable or unitary. However, the contextual and shifting nature of children's ideas in no way takes away from their significance.

Lesson 12 in First Grade: The Two Sisters' Ages ($y = x + 3$)

With the backdrop of the mid-interviews, we turn our attention to Lesson 12, which centered on the same function structure as the mid-interview ($y = x + 1$). Lesson 12 focused on the following situation:

Janice and Keisha are sisters. Keisha is three years older than Janice.

The function underlying the situation can be represented algebraically as $y = x + 3$, where x represents Janice's age and y represents Keisha's age. As shown in Figure 1, during this cycle of the teaching experiment we had been working on functions of the form $y = x + b$. Appendix A shows the situations and underlying functions for the previous three lessons, which focused on functions with the same structure.

As we did in every classroom lesson, we started out the lesson recalling the scenarios on which the previous week's lessons—10 and 11 (see Appendix A)—had focused. When Lesson 10 was mentioned, Bárbara, the teacher/researcher for the lesson, reminded the students that this was the lesson in which she had shared how a second grade student had used $Z + 2$ to represent the number of candies one of the characters in the scenario had. Subsequently, Bárbara introduced the Lesson 12 problem context of Janice and her older sister Keisha.

Working on the Problem with Numbers

After the review, children were asked if they had questions about the problem or if there was anything they wanted to share about the problem. One of the students asked, "How old is Janice?" In a traditional mathematics exploration, the students would be thinking about completing a computational procedure, and they would need to know not only the difference between the sisters' ages, but also the age of one of the sisters. Knowing Janice's age would allow the students to carry out the computational procedure. Zyla's response to this question was, "it could be any number because you didn't tell us yet." At this moment, Zyla recognized that since Janice's age had not been specified, it might take on any of many possible values.

This statement opened the door for children to share their thoughts about how old Janice could be. Sensible possibilities were shared, including 4, 5, 6, 10, 16, and 20. Additionally, some students worked to figure out Keisha's age based on one of these possible ages for Janice. Bárbara redirected the discussion to the indeterminate case by suggesting, "I want a way that could represent every single number. Are there different ways of showing that it's any number?" Zyla, very reasonably given what might be construed as a strange request, said, "We have to wait until you tell us." Bárbara's intent had been to suggest the need for a symbol other than numbers (such as variable notation) that could be used to represent "every single number." When Bárbara suggested she was not going to tell Janice's age, Zyla, again, very reasonably, tossed her hands up and said, "Well, then, we'll have to figure it out."

After the children discussed their basic understandings of the problem, they were given handouts to continue exploring the task at their small group tables. Two handouts were presented. The first handout included the following statements, each separated with blank space for the children to work:

Organize your data about Janice's age and Keisha's age.
Find a relationship between Janice's age and Keisha's age. Describe your relationship.

The first handout was read aloud to the whole class, and the research team and research teacher circulated to help students read the handouts. The second handout, on which only a few students had time to work, included the following statements with space in-between each statement for children to show their thinking:

Write a rule in your own words that represents how Keisha's age relates to Janice's age.
Write a rule using your variables[9] *to represent this relationship.*

Following their small group work, students joined their peers in a large circle. Bárbara set up the function table with the letters J (for Janice's age) and K (for Keisha's age) for each column label and with the values 0–5 in the left hand column (see Figure 7). Since tables had been one of the representations used regularly in the classroom from the beginning of our teaching experiment, the students were fluent with this representation at this point, including the inclusion of letters (e.g., J, K) in tables.

For the case when Janice is zero years old, Antonio first suggested that Keisha would be 10 years old, and then explained that he thought of it because "I wrote it down on my paper." Another student suggested that Keisha would be 25 years old. At this point, students did not seem to understand the quantitative relationship underlying the problem. Tia then disagreed and said that Keisha had to be three years old, saying that "she [Janice] is a newborn" and that Keisha is "one, two, three," counting on her fingers. Bárbara asked if she had added three to zero to figure this out. Tia confirmed and Bárbara added the notation +3 next to 0 on the table.

[9]The word "variable" had been used in the classroom before as shorthand for "variable notation."

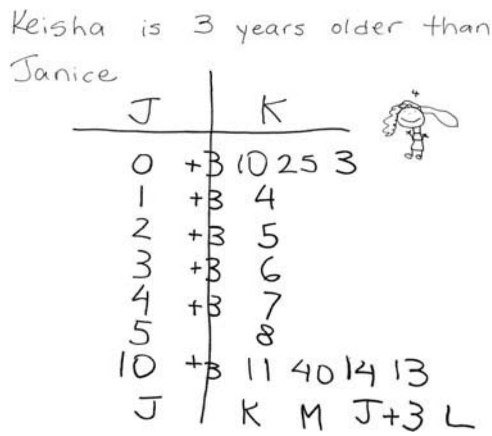

FIGURE 7 Table produced by the teacher, with students' input, on a flip chart at the front of the class.

After this, students suggested the values for Keisha's age when Janice was one, two, and three years old. In each case, when they explained how they arrived at their response, children said that they added three more to Janice's age to arrive at Keisha's age. When Bárbara asked the children, "Why are we always adding three?" Nyah responded, "Because Keisha is three years older than Janice." Then, the students worked on the case when Janice was four years old, using the same approach. For the case when Janice was five years old, Sarah stated that Keisha is eight years old. However, this time, instead of adding three to five, she said she had added one to seven (the output value in the row above in the table, for the case when Janice is four years old and Keisha is seven years old, see Figure 7). This represented a shift in orientation towards the data in that Sarah was not thinking about co-varying values as a function of the problem context (that is, she was not thinking of the two sisters and the fact that they are three years apart in age as a way to determine Keisha's age). Instead, she was reasoning recursively about the values in the output column in the table.

Students faced a new challenge when Bárbara added the case of Janice's age being 10 years. Perhaps swayed by Sarah's recent explanation, the first student suggested 11 years, which is "one more" than 10 years. A second student disagreed and said, "it has to be three more" and suggested 40 years (counting by tens instead of by ones). Other students disagreed. Zyla miscounted three more and suggested 14 years, and Bárbara proposed that the class do it together. When the group counted three more from ten, ". . . ten, eleven, twelve, thirteen . . ." there was a chorus of bobbing heads and looking at neighbors—a consensus that thirteen was indeed three more than ten.

Working on the Problem with Variable Notation

Then, Bárbara addressed the last question on the second handout that asked the children to use variable notation to represent the relationship between Janice's age and Keisha's age. The following conversation ensued:

C1: Researcher/Teacher: So . . . what if Janice is *J* years old, 'cause I never told you how old Janice is. What if Janice is *J* years old [writing *J* in Janice's column in the table]? What should we put right next to it [on the right hand column on the table]? If Janice is *J*, how old is Keisha? Put your hand up if you have an idea that you want to share. Leo.
C2: Leo: *K*.
C3: Researcher/Teacher: *K*? Can you explain your idea?
C4: Leo: 'Cause it— 'cause up [pointing to the columns' labels on the flip chart] there, *J* and *K*.
C5: Researcher/Teacher: Oh, *J* and *K*.

Here, Leo may have been using the variable notation as a label to stand for the name of the sister whose age is being represented. Or it may be (not necessarily to the exclusion of the former possibility) that he was mimicking Bárbara's move: that is, if she used *J* as the label for Janice's column and to stand for Janice's age, it made sense for him to use *K* to stand for Keisha's age. When Bárbara used *J*, she was of course using it to represent a quantity—a quantity of years; it is not clear whether Leo was likewise using *K* to represent a quantity of years, because his thinking was not probed further at the time. What Leo did here was very different from what he had done in his mid-interview, where he had used the order of letters in the alphabet to make decisions about how to express the outcome of the operation, even though he left the independent variable's value indeterminate. If he had continued to use his strategy from the interview, we would expect him to use *M* to represent Keisha's age.

The conversation continued after Leo made his suggestion:

C6: Researcher/Teacher: Okay. Another idea, different idea from this one. Nyah.
C7: Nyah: It can't be *K*, because since we do three, it would be *M*.
C8: Researcher/Teacher: *M*! Okay, so Nyah is saying she has to be *M*. Why does she have to be *M*, Nyah ? . . . What were you thinking, Nyah?
C9: Nyah: Because I see on the alphabet *J* and *K* [looking up at the alphabet chart hung around the wall of the classroom]. There's not three letters between *J* and *K*.
C10: Researcher/Teacher: So what did you do to *J* to get to *M*?
C11: Nyah: I— I went on [points to the alphabet chart displayed along wall, gestures to indicate "going on" by drawing circles with her index finger].
C12: Researcher/Teacher: On? How many?
C13: Nyah: Three.
C14: Researcher/Teacher: Three.

Nyah objected to Leo's suggestion because the relationship between *J* and *K* did not reflect the "plus 3" relationship that had been established between Janice's and Keisha's ages. The fact that Nyah objected and underscored the need to represent the "plus 3" in some way reflects that she had implicitly generalized the relationship between these quantities. According to Nyah's rationale, in order to represent the sisters' ages, the letters used must be chosen so as to encapsulate the relationship between the ages. Using the ordinal relationships between the letters in the alphabet, she established that Keisha's age needed to be "three more" than *J*, and she "went on" from *J* three more letters until she arrived at *M*. Nyah was not looking to specify the age that *J* or *M* stood for, but seems to have felt that the mathematical relationship in question did need to be specified by the letters chosen. The understanding that Nyah revealed about variable notation in this episode is similar to that which she expressed in her mid-interview in which she used the order of letters in the alphabet to make decisions about how to express the outcome of the operation, even though the independent variable's value was left indeterminate.

The conversation in the class continued:

C15: Researcher/Teacher: Another idea? Any other idea for a different way of showing it? What should I put right there? There is *K* as an idea, *M* as an idea. . . . Zyla?
C16: Zyla: I think *J* plus two— I mean three.
C17: Researcher/Teacher: Okay, so Zyla thinks *J* plus three. Could you explain to us your thinking, Zyla? . . .
C18: Zyla: Because, it's— it's just like the second-grader, but it's a different letter.
C19: Researcher/Teacher: Oh, could you remind us of the second-grader? Yeah, I told you about [the second grader] last week, right?
C20: Zyla: Yeah. It was—
C21: Researcher/Teacher: *Z* . . .
C22: Zyla: *Z* plus two.
C23: Researcher/Teacher: For the candy boxes, right [in reference to Lesson 10, see Appendix A]?
C24: Zyla: But this time, this time it— I just said *J* plus three because, um, because, she's three month— she's three years older.
C25: Researcher/Teacher: Oh. Did you hear that? . . . Zyla remembers that I shared with you last week about [the second grader] . . . And [the second grader] said that the candy boxes would be Mary would have *Z* plus two, and so Zyla is saying, "Well, I'm using that idea, and I'm saying then Keisha has to be *J* plus three." Which I think is the same thing that you were saying, Nyah. It's like what you were saying a little differently. Right?
C26: Nyah: [Nods.]

Zyla's suggestion astutely built on the $Z + 2$ expression mentioned in Lesson 10 (see Appendix A). In this lesson in the second-grade class, a student had expressed the number of candies Mary had as $Z + 2$, given that John had Z candies and Mary had two more than John. Using this as a starting point, Zyla suggested that if Janice's age was expressed as J, then Keisha's had to be represented as $J + 3$, "because she's three years older." Like Nyah, Zyla was looking for a way to express the co-varying relationship between Janice's and Keisha's ages. This highlights how relevant and important the use of variable notation could be to help shift children toward working with indeterminate, and ultimately varying, quantities. In this sense, the notation itself could serve as a mediating tool (e.g., Kaput et al., 2008) that could facilitate the emergence of particular kinds of reasoning like generalizing a co-varying relationship.

In this episode, Zyla seemed to have shifted from her earlier use of variable notation in the mid-interview. Whereas in that interview she had expressed reluctance to combine letters and numbers into a single expression ("It's just lame," see line Z22), by this point she seemed comfortable doing so. Moreover, when Bárbara had mentioned to the first graders the second grade student's suggestion of the $Z + 2$ notation the previous week, she had not prescribed it as a "correct" expression. She had simply related that a second grader had suggested this notation, and had asked the first grade class to "think about it," stating explicitly that they didn't have to agree with it. Nonetheless, Zyla adopted and adapted it to reflect the problem being discussed in Lesson 12, altering both the letter *Z* and the number 2 accordingly (to *J* and 3 respectively).

We claim that Zyla's response illustrates that she is able to act on a mathematical expression that includes variable notation as a mathematical object—she takes the second grader's expression of a "plus two" relationship represented as $Z + 2$ and transforms it into a "plus three" relationship by suggesting $J + 3$ to represent Keisha's age in relation to Janice's age. We use the term mathematical object in the way that others have (e.g., Ayers, Davis, Dubinsky, & Lewin, 1988; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky, 1991a, 1991b; Dubinsky & Harel, 1992; Sfard, 2000), alluding to the fact that the learner is able to perform an action on an expression (in Zyla's case, $Z + 2$) that will transform the expression (in Zyla'a case, into $J + 3$). Moreover, acting on the second grader's expression as an object seems to be natural to Zyla, as suggested by the fact that she says "just," as in "I just said J plus three" (see line C24). It is possible that Zyla was not acting on the first expression ($Z + 2$) as an object, and that instead she was exchanging the elements in one expression and situation (Z and 2) with the elements from the other situation (J and 3). Even if this was the case, we can still claim that Zyla was using variable notation to construct an expression that represents the quantities involved in the situation, leaving the quantity that J represents indeterminate.

Meanwhile, Tia had been vigorously shaking her head during Leo's, Nyah's, and Zyla's contributions:

C27: Researcher/Teacher: Okay. You are disagreeing, Tia.
C28: Tia: Because I think it's— I think it's *L* because— because J, K, L.
C29: Researcher/Teacher: Okay. I'll put it down here as your idea. I think that—
C30: Tia: Because—
C31: Researcher/Teacher: I think you're counting on from different places; that's why you're getting different letters.
C32: Tia: J, K, L, M, N, O, P.
C33: Researcher/Teacher: I think you're counting from J, K, L . . . and I think Nyah is doing K, L, M. You're just starting in different places.

Tia's approach was similar to Nyah's: you have to "add on" three letters to J in order to determine Keisha's age. Like Nyah and Zyla, Tia was also focused on trying to express the co-varying relationship between the sisters' ages. Tia counted "three more" letters than J by counting three "fence posts" starting at J (see Carraher, Schliemann, Brizuela, & Earnest, 2006) instead of three intervals starting at J.

The previous episode illustrates the coexistence of a range of understandings within the same classroom. We see in this episode first of all that the four children we focused on were willing to leave the quantity related to the independent variable (in this case, J) indeterminate. We do not have evidence that they were thinking of the quantity as generalized or varying, but they also did not think of quantities as needing to be specific or fixed. It is unclear from the evidence we have whether Zyla's suggestion of $J + 3$ illustrates that she was thinking of the letter used as a generalized number or as a variable, but we can say that J remains an indeterminate quantity. We find this remarkable given prior evidence of older students' inclination to assign a specific value to a letter in an algebraic expression (e.g., Blanton et al., 2015; Carraher et al., 2008; Knuth et al., 2011; Küchemann, 1981; MacGregor & Stacey, 1997). Zyla seems also to illustrate that she may be able to act on a mathematical expression as a mathematical object. In Nyah's and Tia's cases, they were trying to represent the quantitative relationship between the sisters' ages, and they resorted to the order of letters in the alphabet, instead of combining letters and numbers in an expression. That is, they used ordinal relationships between letters in the alphabet to represent quantitative relationships.

DISCUSSION

Through our analyses of Nyah's, Zyla's, Tia's, and Leo's mid-interviews and of the episode from Lesson 12 in our teaching experiment, we have provided evidence that first grade children, of approximately six years of age, can develop a variety of understandings about variable notation. The understandings that we observed in these young children included (1) that variable notation can signify a label or object; (2) that variable notation can represent an indeterminate quantity; (3) that quantitative relationships can be expressed through the ordinal relationships between letters in the alphabet; and (4) that the inclusion of both letters and numbers in a single equation should be avoided. We also observed that these children were able to act on a mathematical expression that includes variable notation as a mathematical object.

We underscore that these children represent a variety of academic achievement levels (according to their teacher, Tia and Leo are in the top third of the class, and Zyla and Nyah are in the middle/bottom third of the class). Moreover, only 8% of third grade students in their school performed above the "needing improvement"/"warning/failing" categories in state mandated standards-based tests the year we carried out our study. We emphasize this because we believe that future studies would find similar understandings among other first grade students from the full range of academic performance.

A compelling finding from our study is illustrated in students' search in their mid-interviews for specific values that could instantiate the quantitative when the discussion did not include variables, juxtaposed with the absence of this search when variable notation was used. This provides support for the claim that introducing variable notation earlier than is typically done could facilitate reflections on indeterminate quantities. The evidence that variable notation could have been acting as a mediating tool (Kaput, 1991; Kaput, Blanton, & Moreno, 2008) that facilitated these kinds of reflections among young students is aligned with our theoretical position that conceptual understandings do not necessarily need to precede the introduction of symbols and that meanings and symbols can co-emerge (e.g., Sfard, 2000) and forms and functions can shift over time (e.g., Saxe, 1999, 2005; Saxe & Esmonde, 2005).

The different understandings that we have identified for variable notation co-occur within the same child, within one interview, as well as among children in the same classroom. We imagine that even older students might draw upon different understandings of variable notation as their understandings emerge and depending on the task and situation at hand, looking for the most efficient ways to express their thinking. As pointed out earlier, the coexistence, lack of stability, shifts, and multiple simultaneous interpretations are consistent with our view of knowledge as context dependent and not necessarily systematic (e.g., diSessa, 1993, 2008; Wagner, 2006). Moreover, because we are not focused exclusively on normative understandings, the variety of understandings espoused by children enriches our descriptions of young children's early algebraic reasoning. In our view, the understandings of variable notation we have identified are all understandings on which children can build.

The first grade children reported on here do not necessarily fully understand what a variable is or that the letters they are using are intended to represent a generalized or varying quantity.

However, we argue that in being able to comfortably and openly use variable notation, and willing to leave the quantities represented by variable notation indeterminate, they are taking critical first steps toward building these understandings, steps that are fundamental in their trajectory towards a fluent use and understanding of variable notation. The abundance of research documenting older students' difficulties with variable and variable notation attests that older students, too, are in fact not—as a developmental perspective sometimes presumes them to be—necessarily "ready" for these concepts.[10] Moreover, by the time they are introduced to these concepts, they have lost important opportunities to construct and extend early algebraic understandings, such as those exhibited by students in the study reported here.

Mason's work (1996) provides an interesting framework for considering the data we have presented. While some might find that we "rushed to symbols" (p. 75) in our early inclusion of variable notation, we argue that we did not given that generalization pervaded the classroom and the explorations of variable notation were focused on meaning-making. We agree with Mason that "letter-arithmetic is in itself insufficient to obtain mastery and smooth transition [to algebra]" (p. 78). However, we argue that allowing variable notation to become part of mathematical language as children generalize and make sense of functions can be potentially powerful. In this sense, the historical misuse of letters in formal, traditional algebra courses for adolescents should not deter us from a productive exploration of these symbols with children. Moreover, Mason's descriptions of the phases in a developing spiral or helix that include manipulating, getting-a-sense-of, and articulating (variable notation) could help to explain children's shifts. For instance, in the case of Zyla's mid-interview, we see her manipulating and getting a sense of variable notation. By the time Lesson 12 takes place, we can see her articulating, reflecting Mason's characterization that,

> as you become articulate, your relationship with the ideas changes; you experience an actual shift in the way you see things, that is, a shift in the form and structure of your attention; what was previously abstract becomes increasingly, confidently manipulable. (p. 82)

Saxe's descriptions regarding the relations between forms and functions are also helpful as we reflect on our study's results (1999, 2005; Saxe & Esmonde, 2005). Using Saxe's framework, we can consider variable notation to be a specific notational form that took on varied cognitive functions for children in this study. As Saxe describes in his work, these functions were not fixed and do not need to be normative. Saxe describes the shifting relations between forms and functions micro-, socio-, and ontogenetically. In our study, we can see how the ways in which children used variable notation, a representational form, shifted over time for each individual child.

The fact that the six-year-olds in this study shifted away from assigning a specific value to a variable and are willing to leave it indeterminate is evidence of the productive nature of their understandings about variable and variable notation. These first grade students are taking their first steps with these concepts and tools—in much the same way that children invent spelling in the process of appropriating written language. Aligned with Cusi, Malara, and Navarra's (2011) ideas about "algebraic babbling" (or the "process of construction/interpretation/refinement of

[10]Only 6% of the 14-year-olds in Küchemann's (1981) study provide evidence of "letter as variable" understandings in his assessment, and in Knuth et al.'s (2011) study, less than 50% of grade 6 students see a literal symbol as representing multiple values. MacGregor and Stacey (1997) report that none of the 11-15 year-old students in their study provided responses on the written test that could be considered this kind of understanding.

'draft' formulas" [p. 492]), we support the design of teaching and learning environments that provide children with extended and sustained periods of time during which they can become fluent and comfortable with variable notation, an essential aspect of mathematical language.

Our study points to several implications for instruction. Postponing the introduction of variables to later years could lead students to use variable notation without meaning. This is consistent with Tall's (1989) position that "our curriculum, designed to present ideas in their logically simplest form, may actually cause cognitive obstacles" (p. 89; see also Sutherland, 1989). Instead, we propose a gradual introduction, from the earliest years of schooling, that encourages the development of productive dispositions towards variable notation. Introducing variables and variable notation earlier than is typical seems feasible and reasonable and may give children opportunities to fully explore their understandings. Simultaneously, it may afford teachers opportunities to engage those understandings explicitly before the stakes are raised, as they are in later grades when noncanonical understandings may be viewed simply as incorrect. It would also allow students a more sustained interaction with a symbol that is known to be problematic for them when they are only given a year or two to appropriate it and become fluent with it.

For the study reported here, we purposely chose a specific slice of data from our larger data corpus. This allowed us to more deeply probe and triangulate students' thinking about variable and variable notation, as exhibited in interviews and classroom discussions, around a specific function type. Our data selection was intended to enrich our current understandings of children's ideas; future studies could describe children's understandings in a broader set of data, to seek representativeness. For example, a wider, more comprehensive, look at trajectories in children's thinking about variable and variable notation, from their initial to their most sophisticated understandings over the expanse of the eight-week teaching experiment, is underway (Blanton et al., 2014). Finally, studies are needed to understand how children conceive of variable and use variable notation in other mathematical domains (Blanton et al., 2011).

ACKNOWLEDGMENTS

The authors wish to thank Brian E. Gravel's contributions to the research here reported.

FUNDING

The research reported in this paper was supported by the National Science Foundation's DRK-12 Award #DRL 1154355. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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APPENDIX A

Situations and Underlying Functions Presented to Students in Cycle 2 of Our Teaching Experiment, Prior to Lesson 12

Lesson #	*Task*	*Function type*
9	Charlotte has a piece of string that she cuts one time. How many pieces of string will she have after she makes the cut? If she cuts her piece of string two times in this way, how many pieces of string will she have after she makes the cuts? If she cuts her piece of string three times in this way, how many pieces of string will she have after she makes the cuts?	$y = x + 1$
10	John and Maria each have a box of candies. The two boxes have exactly the same number of candies in them. Maria's box has two extra candies on top of it. What can you say about how many candies John has? How about Maria?	$y = x + 2$
11	Sara has a jar where she keeps all of her pennies. One day, she finds three pennies on the sidewalk and takes them home. How many pennies does Sara have in the jar? If Sara started with one penny in her penny jar, how many would she have altogether? If she started with two pennies in her penny jar, how many would she have with the pennies she found? If she started with three? If she started with four?	$y = x + 3$