VARIABLES IN ELEMENTARY MATHEMATICS EDUCATION

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ABSTRACT

In this article, I analyze episodes from two third-grade classrooms drawn from a larger classroom teaching experiment to explore how these students began to incorporate nonnumerical symbols in their mathematical expressions when asked to represent indeterminate quantities. The article addresses two research questions: What understandings did these third-grade students construct when they used nonnumerical symbols to represent indeterminate quantities, and how did these understandings vary during the course of working on a single task? What were some of the challenges these third-grade students faced when they first used nonnumerical symbols to represent indeterminate quantities, and how did these challenges vary while working on the Candy Boxes task? Using the constructs of semantic space and form/function relationships, I argue that teaching and learning environments that encourage children's use of nonnumerical symbols, such as variable notation, to represent indeterminate quantities can support children's construction of understandings of variables.

THE role that variables and variable notation[1] should (or could) play in elementary mathematics education is not without controversy. Often times, even in classrooms that encourage children to generalize mathematical relationships, children are only very gradually allowed to interact with variable notation. A frequent approach is to wait until they are in middle school and

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other mathematical understandings are more fully developed before variables and variable notation are introduced.

A delay in children's interaction with variables is often justified by alluding to "developmental readiness" and to narrow interpretations of Piaget's work (for an in-depth discussion, see Schliemann, Carraher, & Brizuela, 2007) that tend to assume that the construction of meanings and/or conceptual understandings must precede the use of symbols or language (for further discussion, see Sutherland, 1989, 1991; Sutherland & Balacheff, 1999). In line with Vygotsky (1978) and other researchers focused on the mediating role of sociocultural tools (e.g., Kaput, 1991), my work assumes that meanings and symbols coemerge (see Sfard, 2000) and that symbols can support the construction of new meanings and conceptual understandings (Saxe, 1999, 2005; Saxe & Esmonde, 2005). From this perspective, the teacher plays an important role in designing tasks and situations through which to introduce variables and variable notation (see Sutherland & Balacheff, 1999).

Some mathematics education research steeped in an early algebra framework (see Carraher & Schliemann, 2007) has been open to exploring variables with students before the middle school years (e.g., Blanton & Kaput, 2005; Blanton, Levi, Crites, & Dougherty, 2011; Brizuela & Earnest, 2008; Russell, Schifter, & Bastable, 2011; Schliemann et al., 2007). Specifically, this research has found that meaningfully using variables is within the reach of young students and that young students can begin to incorporate variable notation into their expressive repertoires, using it to represent, for example, general relationships between quantities. This article builds on past research that adopts a functional approach to early algebra (see Carraher & Schliemann, 2007) and that has already provided evidence of the feasibility of young children using variables in a meaningful way. In addition, it builds on earlier descriptions of the Candy Boxes task (see Carraher, Schliemann, & Schwartz, 2008, where this task and some overall results are mentioned), the task this article focuses on. This article's specific contribution is to provide a detailed, thick description of two third-grade classrooms when nonnumerical symbols were introduced, focusing on the variety of children's ideas. The goal is to describe in detail how this introduction evolved to provide readers with a sense of some of the kinds of challenges students may face, as well as ways in which their ideas might vary during an activity. This remains an important contribution to share with the wider education community given the fact that variables and variable notation remain absent from most elementary school classrooms in spite of ongoing calls to the contrary within the narrower community of early algebra researchers.

Studies carried out with middle and high school students have described the challenges they face when confronted with variable notation for the first time (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984; Filloy & Rojano, 1989; Kieran, 1985, 1989; Küchemann, 1981; Steinberg, Sleeman, & Ktorza, 1990; Vergnaud, 1985). For example, Knuth, Alibali, McNeil, Weinberg, and Stephens (2011), Küchemann (1981), and MacGregor and Stacey (1997) describe that middle and high school students hold, among others, the following views about letters in mathematical expressions (i.e., variable notation): they are something that needs to be evaluated; they are equal to one unless otherwise specified; they are not allowed in mathematics; they are an object/label; they have an alphabetical value; they are a specific unknown, and each unknown is represented with a different letter. In contrast, studies have shown that middle school students whose elementary mathematics experiences included the use of variables and variable notation outperform their school peers who have not had these experiences on test items that involve functions and equations (Brizuela, Martinez, & Cayton-Hodges, 2013; Schliemann, Carraher, & Brizuela, 2012). Experiences with and reflections on variable notation contribute at least in part to these results. Similar results have also been found in other studies that have shown that students who have participated in early algebra teaching experiments significantly outperform control students in their understanding and use of variable notation (e.g., Blanton et al., 2015; Carraher et al., 2008; Kaput, Carraher, & Blanton, 2008).

In the study from which data for this article were drawn, we used indeterminate quantities to leverage students' use of nonnumerical symbols and exploration of variable quantities. According to Radford (2011), "What characterizes thinking as algebraic is that it deals with *indeterminate* quantities conceived of in *analytic* ways. In other words, you consider the indeterminate quantities (e.g., unknowns or variables) as if they were known and carry out calculations with them as if they were known numbers" (p. 310).

In this article, I analyze episodes from two third-grade classrooms that were drawn from a larger classroom teaching experiment to analyze how these elementary school students began to incorporate nonnumerical symbols into their mathematical expressions when asked to represent indeterminate quantities. Within the context of a thick description of the third-grade students in our study as they worked on the Candy Boxes task I address the following research questions: (1) What understandings did these third-grade students construct when they used nonnumerical symbols to represent indeterminate quantities, and how did these understandings vary during the course of working on a single task? (2) What were some of the challenges these third-grade students faced when they first used nonnumerical symbols to represent indeterminate quantities, and how did these challenges vary while working on the Candy Boxes task?

In addressing these research questions, I argue that teaching and learning environments that encourage children's use of nonnumerical symbols, such as variable notation, to represent indeterminate quantities can support children's construction of understandings of variables. In addition, I will argue that from the early days of schooling, it is powerful to include nonnumerical symbols as part of the symbolic repertoire that children interact with in mathematical environments on a regular basis.

Background

Elementary school students and adolescents use nonnumerical symbols to represent variable quantities in ways that differ from the ways in which adults use these symbols (e.g., Brizuela, Blanton, Gardiner, Newman-Owens, & Sawrey, 2015; Brizuela, Blanton, Sawrey, Newman-Owens, & Gardiner, 2015; Küchemann, 1981). Much in the same way that young children invent spelling (e.g., Ferreiro & Teberosky, 1979) and have their own ways of representing numbers (e.g., Scheuer, Sinclair, Merlo De Rivas, & Tièche-Christinat, 2000; Seron & Fayol, 1994), young children use these symbols in different ways, associate different understandings with them, and invent their own symbols.

The conceptual framing for this study borrows from Sfard's (e.g., 2000) and Saxe's work (e.g., Saxe, 1999, 2005; Saxe & Esmonde, 2005). Sfard's and Saxe's work is helpful in framing this study because of the way in which they explain the relationships between symbols, their functions (for Saxe), and meanings (for Sfard) as dynamic and changing. Their work provides a particularly compelling way through which to understand and explain young students' gradual appropriation and understanding of variables and variable notation. Sfard (2000) states, regarding mathematical symbols, "Instead of asking 'What comes first?' [symbols or meanings] we would rather give thought to the question of how to orchestrate and facilitate the back-and-forth movement between symbols and meanings" (p. 92). As Sfard (2000) explains, when a new symbol (such as variable notation) is introduced, "this act of introduction creates a 'semantic space' yet to be filled with meaning" (p. 58). Sfard's work suggests that instead of engaging in this chicken-and-egg debate, we should design "instructional interventions" that encourage a back and forth between symbols and meanings. That is, instead of waiting until children are "ready" and "meanings" for symbols have been established to allow them to interact with new symbols, we should embed new symbols in teaching and learning environments, allow time and exploration for a gradual shift in the meanings that might be connected to these new symbols, and allow symbols and meanings to coevolve. This is exactly the approach we took in the project from which the data for this article are drawn. Saxe's research on the relations between forms and functions (1999, 2005; Saxe & Esmonde, 2005) also helps to frame this study. With this framing, we can consider variable notation to be a specific notational form that could take on varied cognitive functions for children. As Saxe describes in his work, these functions are not fixed or normative.

Through the analysis I present in this article, I will describe the understandings that elementary school students construct about nonnumerical symbols used to represent indeterminate quantities as well as some of the potential challenges they might face in the process.

Study

Setting

The episodes I describe below are drawn from a larger project whose participants included 69 children distributed across four classrooms with whom we worked from the middle of second grade (about 7 years of age) to the end of fourth grade in elementary school. In each of these classrooms, we implemented 90-minute early algebra activities for a total of about 12 to 16 activities each school year. The school in which we were working is a large public school in an urban community, in which 75% of the students were Latino, and where over 83% of the students were eligible for free or reduced-price lunch.

As researchers, we took on the role of instructors during these activities, with the lead classroom teachers providing us with input regarding the areas of elementary mathematics they wanted us to highlight (negative numbers, fractions, etc.). At times, the lead classroom teachers provided some instructional support; however, they were mostly observers during our classroom teaching experiment. All classes were videotaped.

The interactions that I will describe below took place during the first 2 weeks in the fall semester when we worked in the third-grade classroom. The children had already worked with us the previous school year while in second grade, once a week during a six-week period in the spring semester. Our activities that semester had focused on an exploration of additive structures. In our work with them in second grade, we had planned to mention and include variable notation (in the form of letters) in our lessons. In practice, we did not do so consistently or explicitly, and the few children who did use letters did so only as labels or shorthand to stand for characters' names in problems. In this article, I will focus on episodes from two classrooms—three episodes from Classroom A and one episode from Classroom B. A different teacher/researcher was the lead instructor for each one of these classes. I was the teacher/researcher in Classroom A. Classroom A had a total of 16 students and Classroom B had a total of 19 students.

Activity Design

The activities we implemented were designed in such a way that we focused on tasks that highlighted the functional and generalizing aspects of arithmetic. The activities purportedly focused on the arithmetic of addition, subtraction, multiplication, division, fractions, ratio, proportion, and negative numbers. Meanwhile, we documented how children dealt with variable quantities, functions, algebraic notation, function tables, graphs, and equations that were embedded in the arithmetic topics. We considered the following principles when we designed our early algebra activities: tasks were open ended enough that we engaged with them for at least one class session of 90 minutes (in the case of the Candy Boxes task I describe below, we engaged for two class sessions of 90 minutes each); students had opportunities to represent their ideas in multiple ways (e.g., through drawings, written and oral language, tables, variable notation, graphs); and we avoided tasks that required computation in order to solve them. In terms of this last point, we instead designed tasks where the emphasis was placed on exploring the *relationships* between indeterminate quantities.

This article focuses on the Candy Boxes task, the first task we implemented in the fall semester in third grade (see Carraher et al., 2008). In the Candy Boxes task, which took place over the course of two classroom periods separated by a week, the following situation was read to the students by the teacher/researcher: "Mary and John each have a box of candies. Each box has the same quantity of candies. Mary has three extra candies on top of her box."

In the teaching experiment we implemented, we intentionally designed the Candy Boxes task to provide no specific information regarding the quantity of candies in the boxes (see Carraher, Schliemann, Brizuela, & Earnest, 2006, for a description of other tasks we implemented with the same third-grade classroom). As such, the situation allowed for multiple entry points. For example, some children would very likely try to "guess" and assign a single specific quantity of candies to the boxes (i.e., a *fixed unknown* quantity), while some might assign a range of specific values to the indeterminate number of candies in the box. However, because the situation does not necessarily constrain what this quantity should be, it could be thought of as a theoretically *varying unknown* quantity even though the boxes can only contain one specific quantity of candies at any one moment (see also Carraher et al., 2008). Whether children think of the quantity of candies in the box as a *fixed* or as a *varying unknown* quantity, this quantity can be represented using a letter or some other non-numerical symbol that could capture the following covarying relationship: whatever quantity of candies John has, Mary will have this quantity plus three more. After reading the above situation, the children were asked, "What can we say about the quantity of candies that John and Mary have?" and "What can we say about the difference between the quantity of candies John and Mary have?"

Data Sources

We videotaped every activity with two video cameras, one focused on the teacher/researcher and another focused on the children. When students were working in small groups or individually, one camera continued to follow the lead teacher/researcher who was working and talking with students. The second camera roamed the classroom, and the researcher behind this camera engaged with children in conversations and impromptu interviews about the work they were producing. In addition to the videos of all activities implemented, we also collected all written work produced by the students (e.g., see Fig. 1) as well as the overhead projection transparencies that were preprinted as part of the activity materials and that were written on by the teacher/researcher and students in the classroom as the activity evolved (e.g., see Fig. 2).

Data Selection

Given the conceptual framework for this article, focused on Sfard's (2000) semantic space and Saxe's (e.g., 1999, 2005; Saxe & Esmonde, 2005) descriptions of

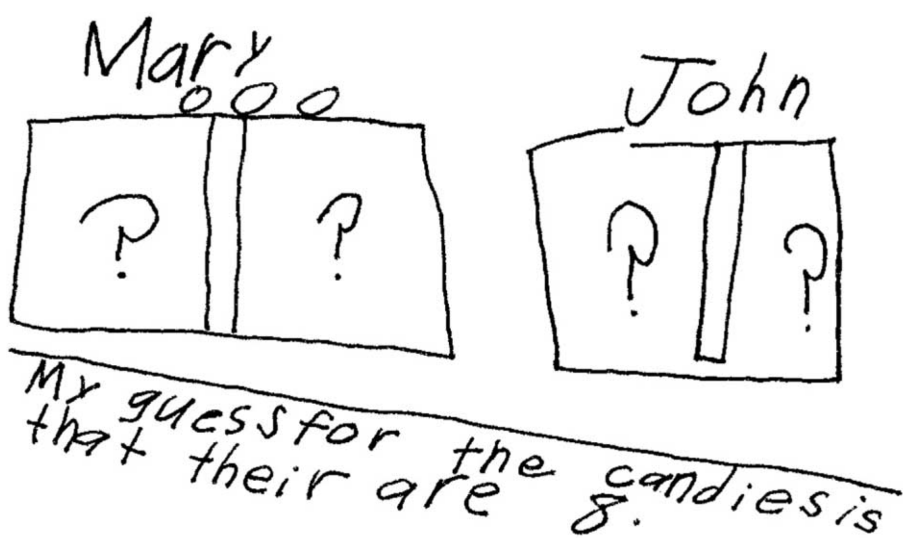

Figure 1. Filipe's work and his proposal ("My guess for the candies is that their [*sic*] are 8").

	John's candies x	Mary's candies z	the difference
Amir	12	15	3
Andy	3	6	3
Anthony	5	8	3
Arianna	3	6	3
Carolina	4	4+3	3
Emily	2	5	3
Filipe	(n) ? 0	? (3)	
James	2	5	3
Jeffrey	3	6	3
Jennifer	4	7	3
Kimberly	8	4/11	3
Max	52	55	3
Nathan	97	100	3
Ramon	3	6	3
Risa	3	6	3
Talik	3	6	3
Filipe	8	11	3
→	5	6/8	3
	3	6	3

Figure 2. Overhead transparency used to document children's proposals as it looked at the end of Episode 2. In Kimberly's row, her first proposal for Mary's candies was four. After discussion with the class, she changed her response to 11. The arrow in the next to last row was a hypothetical case discussed in class, in which students in another class had said that if John had five candies then Mary had six candies. After discussion, the students decided that if John had five candies then Mary had to have eight candies.

the relationships between forms and functions, the data selection first centered on looking for activities when nonnumerical symbols to represent indeterminate quantities were introduced. The rationale for this choice is that I was looking for the opening of a semantic space (Sfard, 2000) that would allow me to describe students' reactions to and explicit explorations of nonnumerical symbols as well as potential variations in the functions for these symbols. This led to selecting the Candy Boxes task. During the implementation of this task, nonnumerical symbols were introduced either by a student (e.g., in Classroom A) or by a teacher/researcher (e.g., in both Classrooms A and B).

The second step in the process involved selecting two (of the four) third-grade classrooms where students were most actively engaged with the task as indicated by their verbal participation. This allowed for close examination of the kinds of conversations that can take place in a classroom when nonnumerical symbols to represent indeterminate quantities are introduced. Similar to the approach described by Nemirovsky, Kelton, and Rhodehamel (2013), the goal in the data selection was "not representativeness but rather the enrichment of the reader's own perception of learning phenomena through immersion in the details of a complex moment" (p. 385).

Analysis

I used Sfard's (2000) construct of semantic space and Saxe's research on form/function relationships as the analytical lenses for the episodes presented in this article. The goal was to identify the understandings children constructed about nonnumerical symbols to represent indeterminate quantities, how these understandings were first articulated, how they evolved, and potential challenges students might have faced. The first step in the analysis involved transcribing the classroom periods in which the Candy Boxes task had been implemented in the two focal classrooms. The second step involved selecting the episodes within the activities that captured the introduction of nonnumerical symbols, the diversity of understandings that children attached to these new symbols, as well as the challenges students faced. This second step required a multimodal approach (Stivers & Sidnell, 2005) that included multiple readings of the transcripts, returning to the videos to capture tone, gestures, and interruptions that might have been lost in the transcripts, annotations to the transcripts to include this additional information, and constant review of all the written work produced during the implementation of the task.

Once the episodes had been selected, I began a line-by-line review of the transcripts (with periodic review of the video clips associated with the transcripts) and marked any moments in the transcripts when a new symbol was introduced and when the students and instructors expressed the understandings they associated with the symbols that had been introduced or any challenges they faced. From the transcripts, I extracted and highlighted the variations in understandings and challenges related to the use of nonnumerical symbols to represent indeterminate quantities in order to answer this article's research questions: *What understandings did these third-grade students construct when they used nonnumerical symbols to represent indeterminate quantities, and how did these understandings vary during the course of working on a single task? What were some of the challenges these third-grade students faced when they first used nonnumerical symbols to represent indeterminate quantities, and how did these challenges vary while working on the Candy Boxes task?* Once these moments had been highlighted, I used them to create a detailed narrative account (or "thick description," see Geertz, 1973) that could help explain the variations in understandings and challenges to describe how the introduction of variable notation evolved in two third-grade classrooms.

In the episodes below, I present extended conversations to provide readers with a sense of the moment-to-moment events in the classrooms. In this way, my goal is to allow readers to develop their own interpretations and to offer my analysis for inspection.

Classroom A

Episode 1: Filipe Uses "?" to Represent an Indeterminate Quantity

The episode began with a discussion regarding how many candies Mary and John could each have. The discussion included the introduction of two boxes, as props, that were meant to stand for John's and Mary's boxes. Both were sealed closed using a rubber band, and Mary's box had three extra candies attached to the top. One student (Jennifer) said we need to know the total number of candies and another (Jeffrey) proposed a specific amount for the content of the boxes. The rest of the class vocally agreed with Talik that the boxes were empty and that therefore John had a total of zero candies and Mary a total of three. Following the discussion, children were provided with paper and asked to show what they knew about the situation. All students in the class provided a single proposal for a pair of numbers that would satisfy the conditions of the problem.[2] Table 1 shows a breakdown of the kinds of work produced by students in Classroom A. Half the students in the class (8/16 students) produced a drawing that included text and/or numbers. One-fourth (4/16 students) of the students in the class produced a nonnumerical symbol. Filipe, who produced question marks, was one of these four students. He produced what can be seen in Figure 1. The other three students used letters as labels to stand for the names of the characters in the problem: *J* for John and *M* for Mary.

As Figure 1 indicates, Filipe used the symbol "?" to represent the quantity of candies in the boxes and circles to represent the three extra candies on Mary's box. It is also possible that Filipe was intending the "?" to represent an uncertainty, and not an unknown quantity. As pointed out below in line 1, however, it is clear that

Table 1. Children's Productions on Their Candy Boxes Handout

What Children Produced in Their Handouts	Classroom A	Classroom B
Used nonnumerical symbols:		
Used *J* and *M* as labels for names and included: drawings and/or numbers and/or text	3 (18.75)	3 (15.79)
Used question marks and included: drawing and text (see Fig. 1)	1 (6.25)	0
Used only text	0	2 (10.53)
Used text and numbers	2 (12.5)	0
Used drawings and included: text and/or numbers	8 (50)	14 (73.68)
Used table	2 (12.5)	0
Total	16 (100)	19 (100)

Note.—Numbers in parentheses are percents.

he was thinking of "how many" candies—that is, of an unknown quantity. The vertical lines in the middle of the boxes are supposed to represent the rubber bands that were used to close the boxes used as props. We might infer that by including "?" on either side of the rubber bands, Filipe was emphasizing that "?" referred to the contents of the entire box, and not just one part of it.

Filipe used a notation he was probably familiar with from his language arts class and other everyday experiences to show that he didn't know how many candies were in the boxes. This notation was neither imposed nor suggested to him. As will become clear in the transcript below, he felt compelled to introduce it given that he was aware of the fact that even his proposed quantity of eight was only hypothetical ("my guess") and that the quantity of candies in the boxes could be any quantity. This highlights the fact that at least some children do potentially have intuitions and resources available to them that they can draw upon to produce these symbols in mathematical contexts. In addition, we should not minimize the potential role played by the constraints and possibilities presented by the specific task the children were working on. Perhaps, precisely because the Candy Boxes task did not ask the children to necessarily engage in computations of specific quantities and did not instantiate the quantities in the candy boxes from the outset, it allowed for exploration of how exactly to represent these indeterminate quantities (Radford, 2011).

Toward the end of this episode, in my role as teacher/researcher, I shared some of the children's productions in order to foster a discussion among the students. When I asked Filipe to share with his peers what he had shown on his handout (see Fig. 1), the following exchange took place:

1 **Filipe:** I put two boxes with question marks 'cause I didn't know how many [candies] were in there. And right there is my theory[3] that was eight [candies in the box].
2 **Teacher/researcher 1 (the instructor for Classroom A):** Okay, now here we have Filipe saying very interesting things. He just talked about a theory. Did you hear that? Did you hear him use that word theory? And he, see he drew two boxes . . . two boxes for Mary, or is that just one box?
3 **Filipe:** Two boxes.
4 **Teacher/researcher 1:** Two boxes.
5 **Filipe:** One box.
6 **Teacher/researcher 1:** One box with two question marks for Mary, because he says, "I don't know how much [candy] is in there." And then the same thing for John, a box with two question marks. And then he drew Mary's three [candies] on top, so he's saying, "I don't know how much [candy] is in there." And then he says he has a theory. And his theory is that there are eight. Where are there eight?
7 **Filipe:** In both boxes, but Mary has three more. So Mary would have ten [miscalculating 8 plus 3].
8 **Teacher/researcher 1:** So Mary would have ten? Are you sure? Eight. What did you do to get ten?
9 **Student:** Eleven.
10 **Teacher/researcher 1:** Eight plus . . . plus three is how much?

11 **Filipe:** Ten.
12 **Teacher/researcher 1:** Does everyone agree with that?
13 **Students:** It's eleven. It's eleven.
14 **Teacher/researcher 1:** It's eleven. See . . . eight, nine, ten, eleven [showing with her fingers]. Right? So Mary has eleven and how many does John have?
15 **Filipe:** Eight.
16 **Teacher/researcher 1:** Eight. Now, see how Filipe used the question marks to show that he doesn't really know how much is in there?
17 **Student:** Yeah.
18 **Teacher/researcher 1:** Would you be willing to accept that maybe there may be ten or zero or one hundred candies in there [in the box]? Maybe?
19 **Filipe:** A hundred.
20 **Teacher/researcher 1:** A hundred could maybe fit in there.
21 **Filipe:** If you could afford them.

In line 21, Filipe mentioned the constraints of the context that could limit the possible domain of values that "?" could take on. In the exchange above, Filipe was willing to accept the fact that we can't actually know how many candies are in the boxes in this scenario, and used "?" to show this indeterminate quantity. In addition, he was willing to accept that the quantity in the candy boxes could vary—he started out calling his proposal of eight a "theory" (line 1), and while he proposed eight on his handout, he also accepted that there could be a hundred candies in the box (line 19), and was even subtle enough to state that this would depend on whether or not you had enough money to afford buying a hundred pieces of candy (line 21). Even though the quantitative relationship between John's and Mary's total candies remained unchanged, Filipe was sensitive to the role and constraints that contexts could play in this relationship. One other noteworthy issue is that Filipe had a relatively hard time dealing with the computation eight plus three, at first insisting it was 10 (line 7). In spite of this being a challenge for him, he was still very clear about the relationship between the quantities, about the fact that his proposal of eight candies in the boxes was just a "theory," and about the fact that the quantity of candies in the boxes remained indeterminate. That is, Filipe illustrates that being able to fluently carry out computations is not necessarily associated with being able to understand and represent general relationships between quantities.

In summary, through the Candy Boxes task, third-grade students in Classroom A began to explore how to represent indeterminate quantities. Filipe proposed "?" since he said we don't know how many candies are in the boxes. He stated that "?" could represent different possible quantities of candies, but also made explicit that even though in principle "?" could represent any possible quantity, there were also constraints to consider, such as how many candies one could actually afford to buy. At this point, there seem to be three coexisting ways to understand and represent the situation. First, the indeterminate quantities could take on different values—for instance, Filipe accepted that the box could have eight or 100 candies, and stated that eight was "just a theory." I claim that he thought of the quantity as potentially *varying*, but *fixed* at any one moment in time. Second, all other students in the class proposed a single pair of values on their handouts, even though during the earlier discussion they had mostly agreed that the boxes had to be empty. Finally, the indeterminate quantities could be represented with a symbol like "?" that could represent myriad values, with some constraints.

Episode 2: Figuring Out How to Represent an Operation on an Indeterminate Quantity

A little later, I was recording students' responses on a two-column table printed on an overhead projection sheet (see Fig. 2). The explicit purpose of the table was to document each student's proposal. The implicit purpose of the table was to highlight that not only were any one of the students' proposals possible, and that thus we were dealing with an indeterminate varying quantity, but that regardless of the fact that the quantity of candies in the boxes was varying, the difference between both covarying quantities was constant.

As described in Episode 1, Filipe's "?" as well as his proposal of eight candies in the box had already been presented to the class. Thus, for Filipe, I included two rows in the table: one with eight candies as the proposal (the third row from the bottom; see Fig. 2), and another with "?" as the proposal (the seventh row from the top; see Fig. 2). As we had done with all the other rows in the table, after including "?" we next turned to what we should write under the column for Mary's quantity if John's quantity was "?."

22 **Teacher/researcher 1:** Remember how Filipe said that he did not know how many was . . . how many there were in John's box and he wrote a question mark [writing a "?" as shown in the seventh row in Fig. 2]? Remember . . . umm . . . Filipe's question mark? Now, if Filipe didn't know how much there were in John's box, what would he write for Mary?
23 **Students:** Three.
24 **Teacher/researcher 1:** Just three?
25 **Max:** Question mark.
26 **Teacher/researcher 1:** Question mark. Now I hear two different things. Some say question mark [adding a "?" in the right column, as shown in the seventh row in Fig. 2]. Some say three [adding a circled 3 in the right column, as shown in the seventh row in Fig. 2]. Do we actually know that Mary has three? What do you think, Filipe?
27 **Filipe:** Three, 'cause . . . three, because if the box was empty and we still have three at the top. It would be empty and we would just put the three inside and you would still have three for Mary.

In line 27, Filipe provided the conditions under which Mary *could* have three candies total: if the boxes were empty.

28 **Teacher/researcher 1:** What, Jennifer?
29 **Jennifer:** But you don't know if there is or isn't in the box [i.e., but you don't know if there are or aren't any candies in the box]. You just have to put a question mark.

In line 29, Jennifer pointed out that we need to represent Mary's candies with a question mark because we can't assume that the boxes are empty. I assume she was using a question mark in this case to represent Mary's total quantity of candies.

30 Teacher/researcher 1: Just a question mark [pointing at the production shown in the seventh row in Fig. 2]?
31 Jeffrey: No, because the three [candies on top of the box].

In line 31, Jeffrey objected to using just a question mark for Mary because we also need to consider the three extra candies that she has on top of the box.

32 Teacher/researcher 1: Now, okay. Now, I hear two things. One is that I should write a question mark [pointing at the production shown in the seventh row in Fig. 2]. Another is that I should write three [pointing at the production shown in the seventh row in Fig. 2]. But if I just write a question mark . . . just a question mark [pointing at the production shown in the seventh row in Fig. 2 and covering the circled 3 shown in this figure] . . . we have two question marks and then . . . someone might think [if] they look at the table that they [Mary and John] have the same amounts. Do the two kids have the same amounts?
33 Students: No.
34 Teacher/researcher 1: No, and if I just write the three [pointing at the production shown in the seventh row in Fig. 2 and covering the "?" shown in this figure] . . . what would it mean if someone saw a three on Mary's side?
35 Nathan: It's three more.

In line 35, Nathan highlighted that Mary has three *more*, not just a total of three.

Some of the children's statements seem to indicate that the children saw the question mark as representing not only John's number of candies but also Mary's total number of candies (i.e., the candies she has *in* the box as well as *on* the box—see Max in line 25 and Jennifer in line 29). This highlights one of the possible constraints of "?" as a representation: there is only *one* of these symbols, so you can't use it to clearly represent different (both John's and Mary's) quantities. In line 32, I was attempting to emphasize that using the same symbol to represent two different quantities (i.e., both the quantity of candies in John's box as well as Mary's total quantity of candies including both the candy box and the three candies on top of the box) would be confusing. Even though I tried to emphasize this, I explicitly decided not to just tell the students that if you use the same symbol more than once it is assumed that they stand for the same value.[4]

36 Teacher/researcher 1: It's three more.
37 James: That's zero for them [in the box] 'cause they don't know. [The three more that Nathan is referring to.] It's three more than zero.

In line 37, James pointed out in what condition Mary *could* have a total of three candies: if the boxes were empty.

38 Teacher/researcher 1: So how could we write that?
39 Nathan: You put a zero in John's candies and leave the three in Mary's [I write a zero in John's column; see the seventh row in Fig. 2].
40 Teacher/researcher 1: But I don't want to write a zero in John's . . .
41 Nathan: Leave the question mark [in John's column].

In lines 39 and 41, Nathan seemed to be saying, "Leave the question mark 'cause that's what you want, but it can still be standing for zero."

42 **Teacher/researcher 1:** I want to leave the question mark.
43 **James:** That [the quantity of candies in the box] equals zero 'cause you don't know what's in it.

In line 43, James seemed to be saying something like, "You can infer nothing about the amount, so nothing is the amount."

44 **Nathan:** And plus the three equals three more.

In line 44 we could infer that Nathan was pointing out that Mary has three more than whatever is in the box (be that quantity "?" or zero).

In the general discussion during Episode 2, in spite of the fact that they had previously proposed specific quantities for the candies in the box, as shown in Figure 2, students shifted and now did not feel inclined to instantiate a nonzero value for the quantity of candies in the boxes while discussing Filipe's proposed symbol "?." The notations included in the right-hand column in Figure 2 represent the fact that students agreed in general on the elements that needed to be part of the expression—three and "?." Some students continued to insist that when we were considering the case of "?" candies for John this was in fact standing for an empty (zero candies) box, as shown in the seventh row in Figure 2 in the left column, but disagreed on how to show three more than "?." Using natural language, Jeffrey (line 31) and Nathan (lines 35 and 44) expressed the idea that Mary's amount needs to be represented as three more than "?" and cannot be represented with just "?." Compared with Episode 1, more students than just Filipe are participating, and all of them are engaging with "?" as a potential symbol that could be used to represent the quantities. In other words, none of them were rejecting the use of a nonnumerical symbol to represent a quantity.

As illustrated in the above exchange, some children were inclined to consider the case of zero candies in the box, thus making three Mary's total quantity of candies. While we don't know the reason for children's inclination, their choice is mathematically relevant. The case of zero candies in the box is an extreme case that highlights the main characteristic of the covariation between the quantity of candies that Mary and John each have: by assigning a zero quantity to the boxes, the transformation (+3) and the constant in the relationship are emphasized. With quantities other than zero for John, that transformation and constant quantity are masked within Mary's quantity. Through their choices, the children themselves emphasized the importance of this constant difference between both quantities that defines the covariation in the relationship, as illustrated when Jennifer made the following statement, at the tail end of this episode: "Everybody has the right answer [no matter what their prediction was] . . . because everybody has three more, always."

At the end of Episode 1, we identified three coexisting ways to understand how to represent the situation—the indeterminate quantities could take on a myriad of values, they could be instantiated by a single pair of values, or they could be represented with a unique, single symbol ("?") that could represent a varying quantity, with some constraints. In Episode 1, because Filipe provided evidence that the boxes could potentially hold eight—which was just a theory according to him—or a hundred candies, I argue that he understood the boxes as containing a *potentially varying* quantity of candies, but that this quantity needed to be *fixed* at any one moment in time (i.e., the boxes either have eight candies, or they have one hundred, but not both). In Episode 2, students saw the quantity of candies in the boxes as potentially *varying but indeterminate*—any one of the quantities proposed was seen as possible, as long as the covarying relationship between Mary's and John's candies was kept constant. At the end of Episode 2, even more ways of understanding how to represent Mary's quantity if John has "?" candies have emerged: Mary has a total of three candies; Mary has a total of "?" candies; the boxes are empty; Mary has "?" candies in the box and three additional candies on top of the box. We can also identify connections between these understandings: children who argued for the boxes being empty also seemed to be trying to justify the case of Mary having a total of three candies. The same situation has elicited among students varied understandings about indeterminate quantities and how to represent them. Understandings have also shifted from Episode 1 to Episode 2.

Episodes 1 and 2 provide us with examples of the coemergence of symbols and meanings and of the semantic space that Sfard (2000) has referred to. The introduction of the new symbol by Filipe was likely anchored in his previous experiences with question marks in written language. It probably connected to students' evolving understandings of and explorations with indeterminate quantities as well as with uncertainty. While Sfard (2000) states that the "act of introduction creates a 'semantic space' yet to be filled with meaning" (p. 58), in Classroom A we see that some understandings for "?" already existed. These understandings varied in the classroom (Saxe, 1999, 2005; Saxe & Esmonde, 2005) as this symbol was used in a new and different (mathematical) context. There was no neat one-to-one mapping between the symbol and a specific understanding. Instead, different understandings coexisted during the classroom discussion.

An open question at this point was how we might expand on Filipe's "?" as the symbol used to represent an indeterminate quantity of candies contained in the box to embed a more conventional representation in an instructional sequence. To this end, toward the end of the episode, I explained to the students that some people used letters in the way that Filipe had used the question mark (see the *n*, the left-most notation in Filipe's seventh row, in Fig. 2).

45 Teacher/researcher 1: Another way to show this, another way to show this just so you know . . . I wanted to let you know, is to write . . . instead of a question mark, it would be the same thing to write an *n*, 'cause you don't know.

46 Nathan and others: That means anything [surprisingly recalling moments during second grade when we had mentioned the use of letters in mathematical expressions].

47 Teacher/researcher 1: So if *n* means anything, 'cause we, what should we write for Mary? If we write *n* for John [writing *n* as seen on the left side of the seventh row in Fig. 2], what should we write for Mary?

48 Nathan and others: Three, just three.

As a research team, we had decided to use *n* in the Candy Boxes task. In subsequent activities, we would use other letters, vary them from activity to activity, and emphasize that what letter we chose was irrelevant. My emphasis in line 45 regarding *n* representing a quantity "you don't know" may have inadvertently focused students on a *fixed unknown* as opposed to a *varying unknown*. Nathan himself rectified this immediately, however, in line 46, by emphasizing that it could be "anything," something already mentioned by other students like Filipe who emphasized that the candy boxes could have different amounts of candies, not a single possible amount, and Jennifer, who stated that "Everybody has the right answer [no matter what their prediction was] . . . because everybody has three more, always."

The students' response regarding how to represent Mary's quantity as a function of John's using *n* was similar to their proposals when using "?." They seemed to move seamlessly from one symbol to the other. Their challenge was not what specific symbol to use (they reacted nonchalantly to *n*) but how to represent one quantity as a function of another one.

Episode 3: Connecting Filipe's "?" to a Letter to Represent an Indeterminate Quantity

The following week, I invited the class to recall the situation we had worked on the week before. These are the things that students remembered: "three is the difference"; "John has three less [candies than Mary]"; "Mary has three more [candies than John]"; "there are the same number of candies in each box." In addition, I reminded students about our conversation about Filipe's "?" and the introduction of *n* as another potential symbol to represent the quantity of candies in the boxes. In this week's activity, I wanted to focus attention on how to express Mary's quantity using *n*, just like we had done the week before with "?." While I had linked Filipe's "?" to *n*, I wanted to explore how to represent Mary's quantity as an operation to be carried out on a nonnumerical symbol (whether that symbol was *n* or "?" or any other).

49 **Teacher/researcher 1:** If we could write what John has in his box in a different way . . . what way could we write it? What's inside John's box? Yeah, Max.
50 **Max:** *n*.
51 **Teacher/researcher 1:** *n*. So we could write *n*. I'll write it right here [in a drawing of John's box on the board]. And if John has *n*, how much does Mary have? . . . If John has *n* . . . um . . . Anthony.
52 **Anthony:** Mary has three.
53 **Teacher/researcher 1:** Does she have only three? How about the ones she has in the box?
54 **Student:** *n*.
55 **Teacher/researcher 1:** That's what she has in the box, Talik?
56 **Talik:** She has . . . she has *n* in the box because if John has *n* and Mary has *n* . . . then that means that there's the same in the box, but John . . . Mary has three more.

57 Teacher/researcher 1: So how can you show that she has *n* and three more?
58 Talik: Because there's three at the top and there's *n* at the bottom for John and Mary.

In line 58, in response to my question, Talik seemed to describe a drawing: *n* in both Mary's and John's boxes, three candies on top of (Mary's) box. It is unclear if Talik was referring to both boxes or only Mary's when he said, "there's three at the top" (see line 58). However, given that he had just stated (see line 56) that Mary had three more, it seems reasonable to assume that he meant "at the top [of Mary's box]" in line 58.

59 Teacher/researcher 1: So how should I write it? What did you say Jennifer? You said anything.
60 Jennifer: Plus anything [perhaps referring to *n* or "?" or any other symbol] plus three.

In this excerpt, Talik and Jennifer provided verbal statements that express how Mary's total quantity of candies could be expressed. Talik said, "She has . . . she has *n* in the box because if John has *n* and Mary has *n* . . . then that means that there's the same in the box, but John . . . Mary has three more" (line 56), and "Because there's three at the top and there's *n* at the bottom for John and Mary" (line 58). Jennifer stated, "Plus anything plus three" (line 60). In addition, it is possible that in line 54, when I asked about Mary, "Does she have only three? How about the ones she has in the box?" and Jennifer answers "*n*," that she is referencing Mary's total amount of candies, even though I thought she was referring to the amount in the boxes only. Even though the students did not produce a conventional written expression for Mary's quantity such as $n + 3$, they were able to articulate what elements the expression needed to include. While I would not claim at this point that all students would be able to produce or fully understand the expression $n + 3$, at least some of the students did provide verbal descriptions of this expression. I suggest that Jennifer's reference to "anything" (line 60) is synonymous to writing *n*, or "?," and, therefore, that her statement "Plus anything plus three" (line 60) could be translated into $n + 3$ or "? + 3." Similarly, in Talik's response, it is clear that he understood that the constant covariation in the situation was that "Mary has three more" (line 56) than John.

A semantic space (Sfard, 2000) that had begun to grow and expand when Filipe introduced the symbol "?" continued to expand with the introduction of another symbol—*n*. In addition, the understandings related to "?" that evolved in Episodes 1 and 2 connected to the new symbol (*n*) that was discussed in Episode 3. Episodes 1–3 provide examples for the coemergence of symbols and meanings (Sfard, 2000): students put forth ideas in relation to the symbols used, and these ideas varied (Saxe, 1999, 2005; Saxe & Esmonde, 2005) in the context of the discussion, the questions asked, and ideas put forth by other students. In addition, the understandings held by students did not necessarily begin to unify in a single direction.

As we see, different understandings for the two new symbols that were introduced—"?" and *n*—coemerged and coexisted. By the end of Episode 3, the students in Classroom A constructed varied understandings for nonnumerical symbols when dealing with indeterminate quantities:

- The same symbol can be used to represent two different indeterminate quantities (e.g., see Max in line 25).
- Indeterminate quantities (represented by "?" or *n*) must be set to zero. In this understanding, the constant difference between both indeterminate quantities (i.e., John's and Mary's candies) is emphasized (e.g., see Anthony in line 52).
- Nonnumerical symbols can be used in an expression to represent an operation on an indeterminate quantity. This can be done:
  - through a drawing (e.g., see Filipe's drawing in Fig. 1);
  - through natural language that describes the scenario (e.g., see Jennifer, line 60, and Talik, line 58).

The understandings held by students shifted from Episode 1 to 3. For example, Jennifer started Episode 1 by saying that we need to know the total number of candies; by Episode 3 she had definitely moved away from this. Students in the class also moved away from trying to find a fixed quantity of candies that were contained in the candy boxes and also shifted from using "?" as a symbol to using *n*.

In terms of potential challenges, we have seen that some students (such as Filipe) may face some difficulties with computations. However, we also saw how this did not prevent Filipe from being able to reason about generalizations and relationships between quantities. In addition, some students tended to overgeneralize the rule that nonnumerical symbols can be used to stand for "any value" and therefore tended to use the same symbol to represent both John's and Mary's candies. Finally, some students tended to assume that the boxes were empty, while still considering the relationship between John's and Mary's boxes of candies. During the Candy Boxes task, we have evidence that children can productively engage in algebraic reasoning. The next subsection focuses on what occurred in another third-grade classroom.

Classroom B

Episode 4: Joseph and Cristian Build Off of the Instructor's Suggestion and Construct an Algebraic Expression That Includes Variable Notation

Episode 4 occurred in a single weekly activity (as mentioned above, the episodes from Classroom A occurred over 2 weeks). In this second classroom, even though the initial situation presented to the students was the same, the conversation evolved quite differently. In this classroom of 19 students, in their handouts, six students provided no specific numbers and instead emphasized the relationship between John's and Mary's amount of candies by only making reference to the difference of three candies between both, one student proposed more than one pair of values for John and Mary, and 12 provided one single pair of values for John and Mary. Table 1 shows the breakdown of the kinds of work produced by students in Classroom B. Three-fourths of the students in the class produced a drawing that included text and/or numbers. Three of the 19 students (15.79%) produced a nonnumerical symbol; in their case, they used *J* and *M* as labels for the names of the characters in the problem.

Episode 4, which focuses on the first week of the Candy Boxes task, began with one of the third graders feeling awkward about proposing any hypothetical quantity for the candies in the box, and Teacher/researcher 2 (the instructor for this class) making a proposal:

61 **Matthew:** Actually well I . . . I well . . . I think without the three maybe it's . . . Yeah, I pretty much don't wanna make a prediction.
62 **Teacher/researcher 2:** Okay, but let me offer you an alternative and see if you're willing to do this. What if I tell you, Matthew, that John has *n* . . . *n* pieces of candy. And *n* can mean any quantity. It could mean nothing. It could mean ninety. It could mean seven. Does that sound okay?
63 **Matthew:** Yeah.
64 **Teacher/researcher 2:** [To Nathan, who was writing down students' predictions on a table like the one shown in Fig. 2.] Alright, so why don't you write down *n*. [To the other students in the class.] He's [Matthew's] willing to accept that suggestion.
65 **Teacher/researcher 2:** Well, now here's the problem, and this is a difficult problem. Matthew, how many should we say that Mary has if John has *n* candies, and *n* can stand for anything?
66 **Cristian:** Oooh. Oooh.
67 **Teacher/researcher 2:** Hold on Cristian. What do you think we should do? How would we call . . . how would we call . . . If this is how many . . . if *n* stands for any quantity that he happens to have . . . okay . . . that John happens to have, then how much would Mary have? You think she'd have . . .
68 **Joseph:** *n*.
69 **Teacher/researcher 2:** *n*?
70 **Joseph:** Yes.
71 **Student:** Yes.
72 **Teacher/researcher 2:** Well, if we write *n* here doesn't that suggest that they have the same quantity?
73 **Students:** No.
74 **Briana:** It could mean anything.
75 **Student:** She could have any quantity like John.
76 **Teacher/researcher 2:** Yeah, it could mean anything. I know just what you mean. But some people would look at it and say it's the same anything if you're calling them both *n*. So how could you mean . . . is Mary supposed to have more than John or less or the same?

In Classroom A, the first proposal for how to represent an indeterminate quantity was brought up by one of the children in the class—Filipe—through "?." In Classroom B, this proposal came from the instructor. Students in Classroom B were introduced to *n* by their instructor to represent "any quantity." Students in Classroom A had been similarly introduced to "?" by a classmate; in their classroom, the nonnumerical symbol seemed to have been understood as representing "anything" and a myriad of possible values. What still eluded children in both classrooms was how to represent one quantity (e.g., Mary's candies) as a function of another one (e.g., John's candies). If *n* is understood to represent any quantity, then using *n* to represent both Mary's and John's quantities is a perfectly reasonable solution. The conversation continued:

77 **Students:** More.
78 **Teacher/researcher 2:** How many more?
79 **Student:** [Mary has] Three more [candies than John].
80 **Teacher/researcher 2:** Three more, so could . . . how could we write down three more than *n* if *n* is what John has? How could we do that?
81 **Joey:** Three . . . cause *n* could stand for nothing.

Joey's reaction and proposal of *n* standing for "nothing" could be understood as similar to Classroom A's proposals of exploring the case with zero candies in the box. It could also be that Joey was stating "*n* could stand for nothing" because *n* is the first letter in the word nothing. But, if this were the case, then he could have also chosen "nine," "nineteen," or "ninety." Given that he did choose "nothing," we have some evidence to support the former interpretation.

82 **Teacher/researcher 2:** It could stand for nothing, but we're telling you that we're gonna use it to stand for any possibility. [To Joey, reassuring him that "nothing" remains one possibility for the quantity of *n*.] Okay, it could stand for nothing. [All this time, Cristian has been waving his hand and saying "Ooh! Ooh!" trying to get the teacher's attention.]
83 **Joseph:** *n* plus three.
84 **Teacher/researcher 2:** *n* plus three?
85 **Joseph:** *n* plus three.
86 **Teacher/researcher 2:** Wow. Explain that to us.
87 **Joseph:** Huh?
88 **Teacher/researcher 2:** Go ahead.
89 **Joseph:** I thought 'cause she could have three more than John. Write *n* plus three 'cause she could have any quantity plus three.
90 **Teacher/researcher 2:** So any quantity plus three. So why don't you write that down: *n* plus three.
91 **Anne [another researcher in the classroom]:** Cristian had his hand up for a long time too. I was wondering how he was thinking about it.
92 **Teacher/researcher 2:** Cristian, you want to explain?
93 **Cristian:** I was . . . I was thinking the same thing.
94 **Teacher/researcher 2:** Go ahead, explain. You think the same thing as Joseph, *n* plus three? Why don't you explain to us your reasoning and let's see if it's just like Joseph's.
95 **Cristian:** Cause if it's any number, like if it's ninety.
96 **Teacher/researcher 2:** Yeah.
97 **Cristian:** You could just, like add three and it'd be ninety-three.

The students in Classroom B did not use their own notation as Filipe had done in Classroom A with "?"; instead, they quickly adopted Teacher/researcher 2's proposal of using *n* to represent indeterminate quantities. Furthermore, Cristian and Joseph used *n* with ease and fluency. Not only did they accept *n*, but they were able to express Mary's candies as a function of John's, as $n + 3$. The proposal for the $n + 3$ notation was made by two different students—Joseph and Cristian—without the need for the classroom instructor to suggest it. In Cristian's case, it is possible that he suggested ninety (see line 95) because it starts with *n*. However, I argue that given how adamant Cristian was that he was thinking the same thing as Joseph, for him ninety was just one instantiation of *n*.

In Classroom B, Teacher/researcher 2's introduction of *n* connected to the students' "semantic spaces" (Sfard, 2000) and allowed the evolution of different understandings related to this symbol and how to use *n* to express covariational relationships. Moreover, some of the understandings that were constructed in Classroom B were quite similar to those constructed in Classroom A around both *n* and "?." For instance, Joey, in Classroom B, claimed that *n* could stand for nothing (line 81), an idea that is similar to that put forth in Classroom A by James (line 37) and Nathan (line 39): that the boxes of candies might be empty. Similarly, Briana, in Classroom B, stated that Mary's quantity of candies should be represented with *n*, since *n* stands for anything (line 74), that is similar to the proposals made by Max (line 25) and Jennifer (line 29) about "?" in Classroom A. While in Classroom B we don't find evidence of understandings shifting for each of the students in the episode like they did in Classroom A, we do find evidence that different understandings coexist in the same classroom. By the end of Episode 4, the students in Classroom B constructed varied understandings for variable notation (*n*) when dealing with indeterminate quantities:

- If a quantity is indeterminate then it should not be fixed (e.g., see Matthew, line 61).
- The same symbol can be used to represent two different indeterminate quantities (e.g., see Joseph, line 68).
- Indeterminate quantities (represented by *n*) must be set to zero. In this understanding, the constant difference between both indeterminate quantities (i.e., John's and Mary's candies) is emphasized (e.g., see Joey, line 81).
- Variable notation can be used in an expression to represent an operation on an indeterminate quantity. This can be done:
  - through natural language (see line 79);
  - through verbalizing an algebraic expression (e.g., see Joseph, line 83).

In terms of potential challenges, just like students in Classroom A, some students in Classroom B tended to overgeneralize the rule that nonnumerical symbols can be used to stand for "any value" and therefore tended to use the same symbol to represent both John's and Mary's candies. Also like students in Classroom A, some students in Classroom B tended to assume that the boxes were empty, while still considering the relationship between John's and Mary's boxes of candies. In both Classrooms A and B, we have evidence that children can engage productively in algebraic reasoning.

Discussion

The goal of the study reported here was to provide a detailed, thick description of two third-grade classrooms in which nonnumerical symbols were introduced—to highlight students' ideas, the challenges they may face, as well as ways in which their ideas might vary during an activity. The data presented add to a growing body of literature that has shown that it is possible to explore the use of nonnumerical symbols and indeterminate quantities in elementary school children's mathematical experiences, even among third-grade, 8-year-old children. As pointed out, studies have shown that students' later algebra understandings and performance are greatly enhanced by early algebra experiences that embed variable notation in early mathematics classrooms (e.g., Brizuela et al., 2013; Schliemann et al., 2012). In a landscape in which variable notation is still not ubiquitous in elementary school classrooms, knowing more, and in more detail, about the first steps in this process remains important.

The episodes in Classrooms A and B built on the same basic principles and responded to specific events in each classroom that encouraged children to make sense of how to represent indeterminate quantities through symbols other than numbers. The study was intended to respond to the following research questions: *What understandings did these third-grade students construct when they used nonnumerical symbols to represent indeterminate quantities, and how did these understandings vary during the course of working on a single task? What were some of the challenges these third-grade students faced when they first used nonnumerical symbols to represent indeterminate quantities, and how did these challenges vary while working on the Candy Boxes task?* In terms of the first research question, the study here presented illustrates the complex semantic space (Sfard, 2000) that is built as children construct understandings for nonnumerical symbols and for indeterminate quantities, a space in which inventive and noncanonical aspects of children's expressions and their use of natural language combine with canonical representations presented in the classroom. Children's understandings vary as they interact with other understandings. These diverse understandings are the building blocks for other understandings to evolve. The episodes presented identified a variety of understandings held by the third graders we worked with. In addition, the episodes from Classroom A also identified the ways in which these understandings shifted throughout the two classroom lessons.

In terms of the second research question, Episodes 1–3 illustrated that the challenge for the children in Classroom A was not whether or not to use "?" or *n* to represent John's quantity of candies. In fact, the introduction of these symbols connected to a productive, constructive, and growing semantic space, where different understandings began to emerge. Rather, the challenges in Classroom A were centered on (*a*) some difficulties with computations; (*b*) how to represent the relationship between Mary's candies and John's candies, or how to represent Mary's candies as a function of John's candies; (*c*) understanding that when the same symbol is used (be it "?" or *n*) it is meant to represent the same quantity or type of quantity; and (*d*) understanding that "?" or *n* can stand for any quantity and not necessarily always for zero.

The challenges identified are not dissimilar to those that have already been described among much older students confronted with variable notation for the first time (e.g., Bednarz, 2001; Bednarz & Janvier, 1996; Booth, 1984; Filloy & Rojano, 1989; Kieran, 1985, 1989; Küchemann, 1981; Steinberg et al., 1990; Vergnaud, 1985). Thus, "waiting longer" does not seem to help students overcome challenges that seem inherent to constructing understandings about the use of nonnumerical symbols, and about variable notation specifically. The challenges identified seem surmountable—not unlike the challenges young children encounter when trying to make sense of other symbols such as written numbers and written language. What the challenges do highlight is that children may need time and space to become familiar with the different uses of this notation, with other nonnumerical symbols in mathematics, and with their different possible meanings. However, the use of a nonnumerical symbol in and of itself is certainly within the reach of these young children—the understandings we see children put forth are all reasonable.

Conclusions, Implications, and Limitations

This study provides evidence in favor of the argument that children need as many opportunities as possible to make sense of and construct their own understandings for a symbol system as ubiquitous to advanced mathematics as variable notation. Children need not use these symbols canonically in order to justify their inclusion in the early mathematics classroom. These early, sustained interactions with nonnumerical symbols, such as variable notation, in mathematical environments are important to create the necessary conditions for students to develop fluency with these kinds of symbols.

The episodes presented provide examples of possible ways in which to leverage students' ideas as they build understandings of variables and indeterminate quantities. In Classroom A, for example, I leveraged Filipe's suggestion of the question mark to explore the representation of indeterminate quantities. In Classroom B, the teacher/researcher took advantage of Matthew's discomfort to propose the use of variable notation. In the context of our study, we are interested in designing learning environments in which children can begin to construct connections between their ideas, intuitions, inclinations, and conventional symbols. Thus, our introduction of variable notation was intentional.

The episodes presented also provide an example of the kinds of tasks that might facilitate the emergence of these kinds of understandings in elementary school classrooms. It focuses discussion on indeterminate quantities and allows children to explore the relationship between quantities, instead of focusing on computation or on solving for a fixed unknown. As we saw in Filipe's case in Classroom A, being able to fluently carry out computations is not necessarily associated with being able to understand and represent general relationships between quantities.

The study does have limitations, not least of which is the very small sample size with which we worked as well as the exploration of children's understandings within a single task. Additionally, other possible instructional experiences could have been designed to allow children to construct understandings for nonnumerical symbols and indeterminate quantities. In Classroom A, for instance, I could have delayed the introduction of the use of the letter *n* to replace "?" and instead focused attention on how to represent Mary's quantity of candies as a function of John's using "?." In Classroom B, Teacher/researcher 2 could have solicited from the children themselves other ways to address Matthew's dilemma—perhaps some student may have suggested the use of a symbol like "?" or some other nonnumerical symbol without this symbol necessarily being canonical variable notation. I argue that all these kinds of experiences are possible and powerful and should be encouraged in early elementary school classrooms.

In closing, our use of indeterminate quantities to leverage students' use of nonnumerical symbols and exploration of varying quantities helps to illustrate possible ways in which to enrich children's mathematical experiences. These experiences provided students with the opportunity to use nonnumerical symbols, including variable notation, as they explored the relationship between two covarying quantities. We should provide students with rich, engaging, and powerful mathematical environments that allow them to make sense of and explore mathematics in all its complexity.

Notes

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1. Throughout this article, when I use the term "variable" I am also implying its notation, and vice versa. I do this simply for communicative purposes, and not because I am conflating one with the other. "Variable notation" is used to refer to letters to represent variable quantities. In this article, when I say variables or *variable quantity* I refer to both varying and fixed unknown quantities (see Blanton, 2008; Blanton, Levi, Crites, & Dougherty, 2011). An example of the former would be x in the expression $x + 7 = y$. An example of the latter would be x in the expression $x + 7 = 10$.

2. As reported in Carraher, Schliemann, and Schwartz (2008), from the larger sample of students from the four classrooms we were working with, 40 of the 63 (63.4%) students who completed worksheets for the Candy Boxes task ascribed a single value to the candies in the boxes and 23 (36.5%) refrained from assigning any specific value to the amounts of candies in the boxes.

3. Even though Filipe used "theory" to mean a "guess," given the focus of the task, as instructor I decided not to emphasize or correct his use of the term.

4. Carpenter, Franke, and Levi (2003) call this "the mathematician's rule for repeated variables" and recommend sharing this rule with students. The reason I did not do so forcefully or explicitly was that, in general, in our teaching we always first spent some time exploring students' ideas and reactions. At this point, as a teacher/researcher, I was still exploring students' understandings.

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