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YOUNG CHILDREN'S NOTATIONS FOR FRACTIONS

ABSTRACT. This paper focuses on the kinds of notations young children make for fractional numbers. The extant literature in the area of fractional numbers acknowledges children's difficulties in conceptualizing fractional numbers. Some of the research suggests possibly delaying an introduction to conventional notations for algorithms and fractions until children have developed a better understanding of fractional numbers (e.g., Hunting, 1987; Sáenz-Ludlow, 1994, 1995). Other research (e.g., Empson, 1999), however, acknowledges the interaction between conceptual understandings and the representations for those understandings. Following the latter line of thinking, this paper argues that children's notational competencies and conceptual understandings are intertwined, addressing the following research questions: (a) What kinds of notations do five and six-year-old children make for fractional numbers?; and (b) What can be learned about young children's learning of fractional numbers by analyzing the notations they make? This paper presents data from interviews carried out with twenty-four children in Kindergarten and first grade (five and six-year-old children), exploring the nuances of their notations and the meanings that they attach to fractions.

KEY WORDS: fractional numbers, notations, representations, young children

1. INTRODUCTION

Four-year-old Nick recently told me, very decidedly, that his age was "four and a half." His reference to "half" was spontaneous, with no prompting or questioning from my part. When I asked him to show his age, four and a half, on a piece of paper, he quickly told me, "actually, I'm just four." Eventually, after telling him that I wanted to see how he was thinking about "half," he made the following notation (see Figure 1).

He then completed this notation by including a five after four and a half. He also included a half before four and after five (see Figure 2).

He read off this second notation as four, four and a half, five, five and a half. In a follow-up interview a few weeks later, Nick made a "number line" (see Figure 3), placing a half between each one of the whole numbers he made a notation for.[1]

If the circular shapes in his notation (see Figure 2) are the halves, we have to read his notation as: half, four, half, five, half. Or: half, four and a half, five and a half. For Nick, perhaps, the circular shape at times comes to mean the half, and at times both the whole and its half.[2] Nick was envisioning

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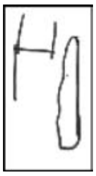

Figure 1. Nick's four and a half.

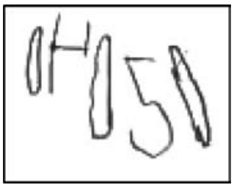

Figure 2. Nick's notations for four and a half and five and a half.

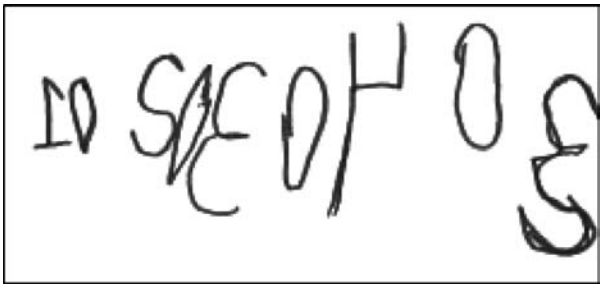

Figure 3. Nick's number line.

halves (already shown to be important in children's development of fractional knowledge; see, for example, Empson, 1999; Kamii, 2000; Piaget et al., 1960; Pothier and Sawada, 1983; Spinillo and Bryant, 1991) as completing a continuous number line (see Lamon, 1999; Siegal and Smith, 1997).

Similarly, Larry, a three-year-old preschooler, recently told me that he was "three and a half" and that on his next birthday he would be four and a half, and then five and a half. Another first grader told me that on his next birthday he would be seven, and on the very next day after his birthday he would be seven and a half. Four-year-old April told me that her sister Sophia was one and she was four and a half, and that, "when I am five Sophia will be one and a half; when I am five and a half Sophia will be two; when I am six Sophia will be two and a half...". For all these children, fractions are an essential part of their daily lives; their references to "halves" are spontaneous and unprompted.

As previous research has found, thinking about fractions assumes a widening of the concept of number. Fractions are "the expression of the part-whole relationship" (Pitkethly and Hunting, 1996, p. 6), a phenomenological source for rational numbers (Freudenthal, 1983), and one of the five sub-constructs that form rational numbers (Kieren, 1980). Kieren (1992), for example, has highlighted the importance of fractions, explaining that they take "a person's idea of number well beyond that of whole numbers" and that they are "a vehicle for relating such number knowledge to many aspects of mathematics and its applications" (p. 327).

Social activities from early childhood, such as sharing and partitioning, provide important foundations for the concept of fraction in informal settings (e.g., Empson, 1995, 1999; Gunderson and Gunderson, 1957; Kieren, 1992; Mack, 1993; Polkinghorne, 1935; Smith, 2002; Steffe and Olive, 1991; Vergnaud, 1983). Researchers have taken this insight even further by studying the importance and role that partitioning plays in children's developing ideas about fractional numbers (e.g., Behr and Bright, 1984; Behr et al., 1983; Empson, 1999; Kieren, 1980, 1992; Piaget et al., 1960; Pothier and Sawada, 1983).

The research on fractional numbers carried out with children in late elementary and middle school has highlighted and explored, especially, the role and use of measurement contexts and number lines (e.g., Behr and Bright, 1984; Behr et al., 1983; Carraher, 1996; Keijzer and Terwel, 2000; Lamon, 1999; Moss and Case, 1999). Siegal and Smith (1997), however, suggested introducing fraction concepts to younger children, at grades two and three, before the use of numbers as mere counting integers could be ingrained, suggesting younger children's ability to conceptualize fractional numbers. As stated before, with younger children, sharing and partitioning has been explored as a context in which children have informal knowledge (e.g., Empson, 1999; Piaget et al., 1960; Pothier and Sawada, 1983). For example, Empson (1999) explored first graders' equal-sharing understandings, their representations for these understandings, and their interpretation of fractional representations, showing that "much younger children can also make sense of fractions through equal sharing" (p. 284). The "representational tools" (p. 289) she focused on were, mostly, part-whole representations. The research reported in this paper was carried out with even younger children, in Kindergarten and first grade, and focusing centrally on their spontaneous notations for fractions. Pothier and Sawada (1983) included children in this age range in their sample of 43 children (their sample extended from Kindergarten to third grade), without focusing on their notations.

Researchers in mathematics education who focus on fractional number learning have wondered about the relationships between conceptual understanding and symbolic competence (e.g., Ball, 1993; Hiebert and Wearne, 1987; Lamon, 1994; Mack, 1990, Sáenz-Ludlow, 1994, 1995). Mack (1990), for example, has observed that among the sixth graders with whom she worked, their "informal knowledge related to comparing fractional quantities was initially disconnected from their knowledge of fraction symbols" (p. 22). Some of the research in the area suggests possibly delaying an introduction to conventional notations for algorithms and fractions until children have developed a better understanding of fractional numbers (e.g., Hunting, 1987; Sáenz-Ludlow, 1994, 1995). However, Empson (1999) proposes that "representational tools" (or notations, as I refer to them), have a "transformative function", and that these tools result "in concepts and processes that in the absence of those tools would not exist" (p. 289; see also Cobb et al., 1997). If we follow this proposition, I argue, we need to explore *both* children's ideas *and* notations for fractional numbers *simultaneously*, as each might highlight different aspects of children's understanding. In response to this position, the research described in this paper addresses the following research questions:

(a) What kinds of notations do five and six-year-old children make for fractional numbers?
(b) What can be learned about young children's learning of fractional numbers by analyzing the notations they make?

In this paper, I shall use the term notations to refer to written numerals or symbols. Notations have also been referred to as external representations (see, for example, Goldin, 1998; Goldin and Shteingold, 2001; Lee and Karmiloff-Smith, 1996; Martí and Pozo, 2000). The research reported in this paper is novel because, although the ideas that children express are similar to those observed in the extant literature (e.g., Ball, 1993; Empson, 1999; Kamii, 2000; Mack, 1990; Pothier and Sawada, 1983; Sáenz-Ludlow, 1994, 1995), these children are younger than most previously studied, and their ideas about fractional numbers were studied *through* the spontaneous notations they made, as opposed to just conceptually. In addition, the children in the sample spontaneously used numerals in their notations, and not just part-whole representations.

2. METHODOLOGY

2.1. *The sample and some preliminary observations*

The interviews that form the corpus of data for this paper were carried out in two Kindergarten (children need to be 5 years old upon entering Kindergarten) and two first grade (children need to be 6 years old upon entering first grade) classrooms at a public school in the Boston area (in the USA). The public school at which the interviews took place can be considered representative of the general population of the Boston area of the USA (31% non-White students, and 54.4% low-income family students, as defined by the Massachusetts Department of Education). Although all Kindergarten and first grade students at the school were invited to participate, only those whose parents responded to the invitation and consented to their participation were included in the sample. This resulted in a sample of seven Kindergarten and seventeen first grade students – a total of twenty-four students.

The children were exposed to a traditional mathematics curriculum, guided by textbook instruction. Only in first grade were they briefly exposed in some way to fractions formally, by way of sharing and partitioning contexts. First graders also had number lines that only marked whole numbers on each one of their desks, as a decoration for their name labels. As is the case for other children, we could assume that they might be exposed to information about fractions through their home environment, or through other media in their social milieu (advertisements, television, printed materials, books, money, etc.).

When asked when they had heard of halves, most of the children interviewed mentioned a sharing or partitioning context. Many of the children were concerned with the "shape" of the fractions (i.e., the shape of the "pieces of" pizza or cookie, or the size and shape of the pieces in the pie chart), and if they were even, or all the same size (many first grade children in Empson's (1999) intervention were also concerned with the "fairness" of the sizes of the fractions). Children in the sample tended to call all fractions "halves," regardless of whether they were halves, thirds, quarters, or fifths, just as Ball (1993) observed among the third graders with whom she worked, and Empson (1999) observed among her first graders. This is also consistent with Pothier and Sawada's (1983) first level of partitioning capabilities among young children (the level of sharing). In addition, some of the children were not even comfortable naming the fractions of pizzas or cookies, and simply referred to them as "pieces" – two pieces (halves), three pieces (thirds), and so on, as Mack (1990) found among the sixth grade children with whom she worked. A couple of children in the sample even spoke of "the line" as being "the half": the partitioning line that separates the pieces in a pizza or a cookie is "the half", not each one of the pieces themselves; "the line" is also the line separating numerators from denominators, as we will later see. Thus, children in the sample exhibited some of the same conceptualizations and behaviors previously observed by other researchers (e.g., Ball, 1993; Empson, 1999; Kamii, 2000; Mack, 1990; Pothier and Sawada, 1983. Note that Ball and Mack worked with older children than those described in this paper: third and sixth graders, respectively).

2.2. *The interviews*

The interviews were carried out as clinical interviews (see Piaget, 1929/1926). According to Piaget, in the clinical interview the interviewer must know how to observe, "that is to say, to let the child talk freely" (1929/1926, p. 9). These individual interviews were described to the children as conversations about "what you know and think about numbers." During the interviews, each child was presented with a series of questions. The questions structured the interviews but also provided the flexibility necessary to a clinical interview. The interviews focused, not on a particular answer I was seeking from the children, but on the thinking and understanding that went into solving the various tasks. Following the clinical interview method, each child was presented with potentially conflicting situations during the course of the interviews, and the contradictions in their justifications or thinking were made explicit and explored.

There were five broad sets of questions asked. The first three (1–3) focused on the children's age (a measurement context), trying to build on their informal knowledge, shown to be an important source for developing fractional number knowledge (e.g., Empson, 1999; Lamon, 1994; Mack, 1990; Sáenz-Ludlow, 1994, 1995). In addition, the first four sets of questions (1–4) focused on "halves", also shown to be a focus of initial fractional understanding (e.g., Ball, 1993; Empson, 1999; Kamii, 2000; Piaget, Inhelder and Szeminska, 1960; Pothier and Sawada, 1983; Spinillo and Bryant, 1991). The last two sets of questions (4–5) focused on a sharing and partitioning context, which provides an important foundation for the concept of fraction (e.g., Empson, 1999; Kieren, 1992; Mack, 1993; Steffe and Olive, 1991; Vergnaud, 1983). The last set of questions (5) focused on quarters: Pothier and Sawada (1983) have shown that after sharing in halves, children go through a second stage in their conceptualization of fractional numbers in which they conceptualize fractions in powers of two. This is consistent with Piaget, Inhelder, and Szeminska's observations (1960), as well as Empson's (1999) and Kamii's (2000). The following is the general template for the questions asked during the interviews:

1. How old are you? Could you show me on this piece of paper how old you are? If the child did not respond with a fractional number (e.g., five and a half, six and a half), they were told: "I was asking some of the other kids how old they were and some of them said [the interviewee's age]

and a half. Have you ever heard of that? Anyone being [the interviewee's age] and a half? What do you think that means? Could you show me on this piece of paper how old they are?"
2. What is a half? What does half mean to you? Could you show me half on this piece of paper? What does it mean to be [the interviewee's age] and a half? Why do they say that you/they're [the interviewee's age] and a half? Is [the interviewee's age] and a half more than [the interviewee's age], less than [the interviewee's age]? How do you know? When is your birthday? When will you turn half?
3. Have you ever heard anyone else talk about halves? Have you heard of halves that are not how old you are? Does anybody talk about halves at home?
4. Imagine we have three cookies and you and I want to share equally so it's totally fair and no one eats more than the other. How much will each of us eat? Can you show me on this piece of paper?
5. Imagine we have five cookies and you want to share the cookies equally among four people so it's totally fair and no one eats more than the other. How much will each of the four people eat? Can you show me on this piece of paper?

The interviews were carried out individually. Each one of the interviews was videotaped and then transcribed verbatim. In addition, all notations made during the interview formed part of the data collected. If the children mentioned a context, situation, or problem other than those brought up, their ideas were followed up. The children were asked to show on paper each one of the ideas they presented and were encouraged to make as many notations as possible related to their understandings, as well as to explain the notations they made.

2.3. *The analysis*

The data from the interviews will be presented in two sections: one dealing with the notations children made for fractions, using numerals, and the second dealing with the meanings that the children attached to the notations they made and to fractions in general. The analysis of the data involved three general stages:

Stage 1: The analysis involved an exhaustive revision of the transcripts and notations produced in the 24 interviews. One salient and surprising observation in the data at this level of the analysis was how many of the children used numerals in their spontaneous notations for fractions.

Stage 2: While the transcripts and notations were being revised, recurrent responses, behaviors, and notations were identified. This level of analysis resulted in identifying three major themes in children's responses: (a) non-conventional notations; (b) conventional notations; and (c) meanings attached to the different notations. In addition, atypical ways of notating or meanings attached to notations were also identified, even if only one child exhibited them.

Stage 3: Once recurrent themes had been identified, the transcripts were once again revised in order to verify the interpretations made. These recurrent and atypical responses, behaviors, and notations became the categories shown in Table I.

3. Children's notations for fractions using numerals

Table I summarizes the types of notations children made for fractions (on the left-most side), as well as the meanings children associated with fractions (on the right-most side of the table).

Of the 24 children interviewed, 17 of them (71%; 4 from Kindergarten and 13 from first grade) made some kind of notation including numerals in their notations.

3.1. *Non-conventional notations*

12 of the 17 children (71%) who used numerals for their notations made non-conventional notations.

3.1.1. *Inclusion of a line*

Most of the children who made notations for fractions included a line in their notation (10 out of 17, or 59%) in a non-conventional way (these ten children do not include those that made conventional notations for fractions who, of course, included lines separating numerators from denominators). Of these 10 children, half of them used only a line, while the other half included some numeral below the line, such as 2, 3, or 0.

3.1.1.1. *Inclusion only of a line as a notation for a fraction*

Examples:

Kindergarten:

Zachary: When asked to make a notation for six and a half, Zachary stressed the line that must be made after the numeral six (see Figures 4 and 5). In Figure 4, the line resembles the one separating numerators from denominators, as in 1/2. In Figure 5, the line bears resemblance to the partitioning line separating parts in wholes.

TABLE I
The types of notations made and the meanings associated with fractions

		Types of notations made using numbers							
		Non-conventional notations Inclusion of a line					Different meanings for fractions		
Grade	Name	Only a line	A number below the line	Use of a plus sign	Crossing out of a number	Conventional notations	Half is a little bit	Order, but no match with time	Match-up of fraction with time and magnitude
K	Haley	–	–	–	–	–		×	
K	Jonathan	–	–	–	–	–		×	
K	Joseph	–	–	–	–	–			×
K	Kassandra					×*			×
K	Michael		×					×	
K	Sheila			×			×		
K	Zachary	×						×	
G1	Adrianna	–	–	–	–	–	×		
G1	Allen	–	–	–	–	–			×
G1	Brenna	×					×		
G1	Chris					×			
G1	Dakota					×		×	
G1	Darky	–	–	–	–	–	×		
G1	Joey					×	×		

(*Continued on next page*)

TABLE I
(*Continued*)

		Types of notations made using numbers					Different meanings for fractions		
		Non-conventional notations Inclusion of a line							
Grade	Name	Only a line	A number below the line	Use of a plus sign	Crossing out of a number	Conventional notations	Half is a little bit	Order, but no match with time	Match-up of fraction with time and magnitude
G1	Jason	–	–	–	–	–	×		
G1	Kristin	×					–	–	–
G1	Lucas		×				–	–	–
G1	Manasse		×						×
G1	Martine		×					×	
G1	Matthew		×				×		
G1	Monique					×	×		
G1	Rebecca	×					–	–	–
G1	Taylor				×	×*	×		
G1	Tianna	×					×		
N	24	5	5	1	1	6	10	6	4

*In some contexts.
"–" mean no response was provided or obtained in this area for these children.

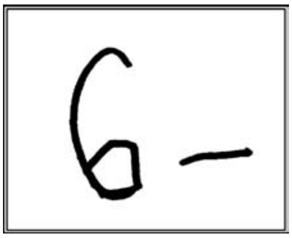

Figure 4. Six and a half.

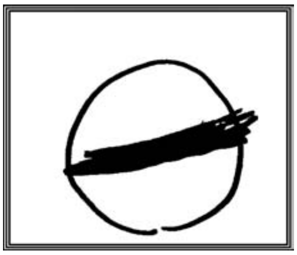

Figure 5. Half (stressing the line as he spoke).

3.1.1.2. *Inclusion of a line and a numeral below the line*

Three of the 5 children (60%) in this group used a two below the line as a notation for half (Ball, 1993; Empson, 1999; Kamii, 2000 and Pothier and Sawada, 1983 have described half as being prevalent among the children with whom they worked). However, just the line and the two below the line could be a notation for "and a half." The other two children in the group used a three in one case and a zero in the other case. One other Kindergarten child, Kassandra, who made conventional notations in some contexts, also used a ten as a denominator in one example (see Section 3.2).

Examples:

First grade:

Lucas: In each of his notations for half, he included the numeral for the whole number, the line below the numeral, and then a two below the line (see, for example, Figures 6 and 7). When he wrote seven and a half as 7/2, I asked him how he knew that he should write a two in seven and a half, and he said, "I know to put a 2 cause one and a half has a 2." He is

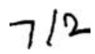

Figure 6. Seven and a half.

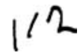

Figure 7. One and a half.

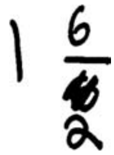

Figure 8. One and a half.

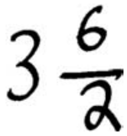

Figure 9. Three and a half.

generalizing from his notation for one and a half, as if he were thinking: "whenever I hear 'half', I must include a line and a two."

Martine: Martine's notation is a variation on that previously shown, in that she includes a six as a numerator for her notations for fractions. In her notations for seven and a half, one and a half, and three and a half (see, for example, Figures 8 and 9), Martine appropriates some of the main features of the conventional notation for half: a line to separate numerator from denominator, a two consistently used as denominator.

3.1.2. *Using a plus sign as a notation for a fraction*
One child in the sample used a plus sign when asked to show fractional numbers.

Example:

Kindergarten:

Sheila: When asked to make a notation for six and a half and seven and a half, she wrote 6+ and 7+, respectively (see Figures 10 and 11). This could lead us to believe that Sheila can think of six and a half as being more than six and of seven and a half as being more than seven. However, she then stated that six and a half is less than six and that on the very same day she would turn seven years old and she would be seven and a half. Perhaps, then, Sheila may be using the plus sign as a stand-in, place-holder, or dummy number (see Alvarado, 2002; see also endnote 2) for "half," and not to show "more." It may be that, for her, half needs to be represented by something;

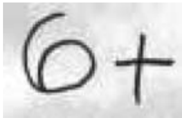

Figure 10. Six and a half.

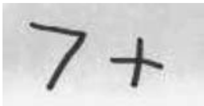

Figure 11. Seven and a half.

there cannot be a half involved without it being represented in some way; she thus uses the plus sign for this purpose.

3.1.3. *Crossing out a numeral as a notation for a fraction*

In crossing out a numeral, Taylor is symbolically capturing the sub-division and partitioning of a number.

Example:

First grade:

Taylor: Taylor expressed her conceptual understanding through her notation: she crossed out the numerals two, six, and seven (see, for example, Figures 12 and 13). She shows each of these numerals cut in half – not necessarily seven *and* a half, or six *and* a half.

However, when asked to show fractions in a partitioning context, Taylor used conventional notations: 1/8 for one eighth, 1/10 for one tenth, 1/2 for half, and 1/4 for fourth (see, for example, Figures 14 and 15).

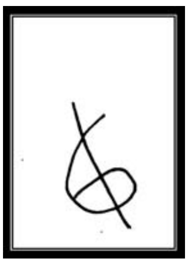

Figure 12. Six and a half – a six crossed out.

Figure 13. Two and a half – a two crossed out.

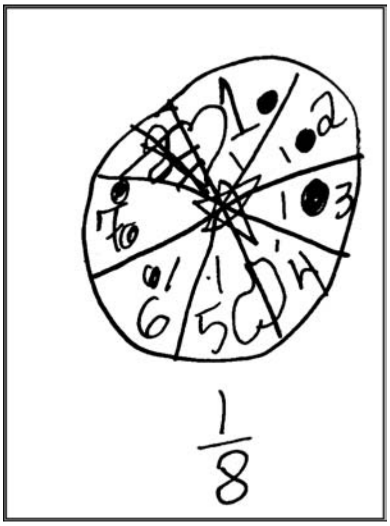

Figure 14. One eighth.

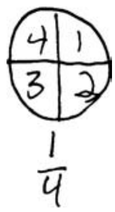

Figure 15. One fourth.

In her striking of numerals in half, Taylor tries to bring together her conception about half with a reasonable notation for it: it is not a whole number, it is a half, so the whole number should be cut in half. As in Kassandra's example from the Kindergarten students in Section 3.2, perhaps Taylor *needs* the partitioning context to be able to use the conventional notations. Her crossing out of the numerals captures graphically the action of halving. But, as pointed out before, she is not really showing, for example, seven *and* a half, but a partitioning of seven into two halves.

3.2. *Conventional notations (at least in some contexts)*

Only one Kindergarten student, Kassandra, used notations for fractions in a conventional way, although not all of the time.

Examples:

Kindergarten:

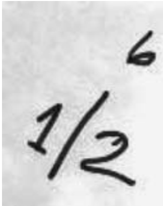

Figure 16. Six and a half.

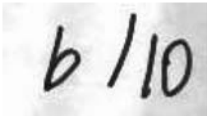

Figure 17. Six and ten quarters (six and three quarters?).

Kassandra: For six and a half, she first wrote 6/12, and then she corrected to 6 1/2 (see Figure 16). She also talked about six and ten quarters (presumably she meant six and three quarters), and she wrote this as 6/10 (see Figure 17), when referring to her current age. In their research, Pothier and Sawada (1983) found that children progress through five stages in their conceptualization of rational number. The first stage entails learning to partition by two, the second entails halving in powers of two, and the third partitioning in even numbers. Piaget, Inhelder, and Szeminska (1960), Empson (1999), and Kamii (2000) have made similar observations. Perhaps, Kassandra's names and notations for six and a half (initially as 6/12), and six and ten quarters (6/10, see Figure 17) can be interpreted within this framework, even though none of the researchers mentioned focused on children's notations using numerals. We could also speculate about Kassandra's knowledge of quarters from a money context, as Ball (2003) indicated among her third graders. Possibly, however, being "six and three (ten)-quarters" old was simply something Kassandra had heard her parents say to her.

When partitioning wholes, she used conventional notations for the parts: 1/4 for quarters (see Figure 18), 1/2 for halves, and 1/3 for thirds (see Figure 19).

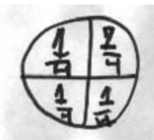

Figure 18. Quarters.

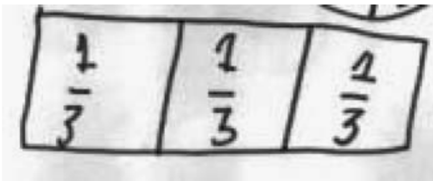

Figure 19. Thirds.

Therefore, when thinking about parts of wholes, Kassandra was quite comfortable using conventional notations. This facility is similar to what was observed in Taylor, the first grade student described in Section 3.1.3. Given these responses, we could speculate about children's understandings and notations for fractional numbers if contexts beyond those of sharing and partitioning were stressed in the school setting.

First grade:

Monique: Monique's notations for half, in seven and a half, six and a half, five and a half, and one half, were some of the few conventional notations made in this group (see Figures 20 and 21). In her case, however, these notations were made in absence of a sharing and partitioning context. Unlike Taylor and Kassandra, described previously, Monique did not need the sharing and partitioning context to be able to make conventional notations.

The students interviewed were comfortable using the different names for fractions, even if these were not always the conventional labels. In general, the children in the sample have assimilated the use of the line separating numerator and denominator, as well as the "proper" names for fractions. Many of the children in the sample spontaneously use numerals in their notations for fractions. Children in the group, some of them still in Kindergarten, show us how they have a way of assimilating the conventions

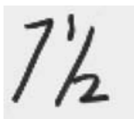

Figure 20. Seven and a half.

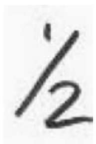

Figure 21. Half.

they are presented with: a slash, a line, the inclusion of a numeral two when referring to a fractional number. While these notations are not always conventional, they do show a partial appropriation of conventional notations for fractions.

4. THE MEANINGS CHILDREN ATTACH TO FRACTIONS AND THEIR NOTATIONS FOR FRACTIONS

As we saw in the previous section, most children in the sample did *not* make conventional notations for fractions (12 of 17 children, or 71%). In spite of this, they have picked up on some of the conventional aspects of fractions, however trivial they may seem: the use of lines to separate numerators from denominators, the inclusion of a numeral two in the place of denominator. Next, we shall turn to see what kinds of *meanings* are mapped on to these ways of making notations for fractions. The right-most section of Table I summarizes the types of meanings children may be interpreted to have constructed for fractions.

While in the types of notations they made many of the children were focusing on trying to figure out what the line meant, and incorporating the line in some way into their notations, in trying to construct some meaning for fractions, most of the children were focused on the fact that half is "a little bit." This meant that for many children, given that half is a little bit, seven and a half needs to be less than seven (see Mack, 1990, for her observation of sixth graders' references to fractions as "pieces"). Fractions are, for children, little bits of numbers.

4.1. *Half is a little bit*

Some children in the sample took to heart the fact that a half is smaller than one and extended that understanding to any context in which they encountered half.

Examples:

First grade:

Taylor: When making a notation for seven and a half, Taylor crossed out the numerals (see Figures 12 and 13). Then, she stated that when she is seven and a half she is not "whole seven," also stating that when she is seven she is older than when she is seven and a half. This actually makes sense in the context of a half seven or a crossed out seven, as she had shown in her notation.

Tianna: When I asked Tianna if she could locate "the halves" on the number line, she stated that she couldn't, because "the halves are the lower numbers so they don't fit on the number line." She implied that halves are smaller numbers that have no location on a number line; fractional numbers are not numbers in and of themselves (see, for example, Pothier and Sawada, 1983, who made similar observations among the youngest children they worked with).

4.2. *Different understandings across different contexts*

There are multiple contexts in which children need to consider fractions. In the interviews carried out, for example, children needed to take into consideration the relative magnitude of fractions, in different contexts, like partitioning contexts (with cookies or pizzas), and measurement of time (with age). The children in this group could make sense of fractions in some contexts but not others. Each one of these contexts highlighted particular aspects of fractional numbers. Carpenter et al. (1993) have highlighted that it is important to broaden informal understandings and children's familiar contexts beyond their local scope. Similarly, Ball (1993) found, in her work with third graders on fractional numbers, the importance of letting them deal with conceptual complexity, and that, "managing a suitable tension between focus and openness in the representational context is crucial" (p. 164). Most importantly, Empson (1999) has highlighted that, "the differences in children's thinking may be due to the affordances of each problem context" (p. 329).

Examples:

Kindergarten:

Haley: Haley explained that seven and a half is more than six, more than six and a half, and more than seven. However, when I then asked her when her birthday was (December 10), and I asked her when she would be seven and when she would be seven and a half, she said that on the same day she would be both seven and seven and a half. Therefore, although she could order the numbers in isolation of any context, she was not able to map this understanding on to a timeline using constant units of measure (days or months or hours). When I asked Haley to show on a number line the different numbers, she showed both six and six and a half at the position of number six on the number line, and she also showed both seven and seven and a half at the position of number seven, mirroring her explanation regarding her birthday. Different contexts highlighted the nuances in her differing and developing understandings of fractions. Had a single context been used, these nuances would not have been captured.

4.3. *Similar understandings across different contexts*

Children in this group are able to think of fractions in different contexts, exhibiting similar understandings and a consistency in their conceptualizations beyond the local scope of each particular context.

Examples:

Kindergarten:

Kassandra: Kassandra explained to me that after six, the next number is "a half." She said that, "a half is a part of almost six." She also said that she was almost seven, so she was six and ten quarters (possibly meaning three quarters). She had time and fractions very neatly matched up in her mind: she would be seven in July (she was interviewed in June when she was "six and ten quarters"), and she would be seven and a half in January. She explained that, "every January I turn halves, and ten quarters [three quarters?] in June." Kassandra was able to make conventional notations for fractions, as well as match up the names for fractions with a timeline.

5. Relations between notations and meaning

The possible relationships between notational and conceptual understandings have been investigated before (see, for example, Brizuela, 2004). The relationship between conceptual understanding and expertise in making notations is complex. It is true that these areas of expertise are related. There are interactions between both areas, and we cannot argue for a full understanding if either conceptual or notational understandings are absent. However, both areas develop and are constructed over time. Monique's example provides an illustration of a first grader who makes conventional notations for fractions. This use of conventional notations might lead us to think that the meanings constructed about fractions are also conventional. As we shall see, this is not always the case.

Examples:

First grade:

Monique: Monique made some of the few conventional notations in this group (see Figures 20 and 21). However, when asked about her age, she explained that when you are seven and a half, "you're not seven but you're a little bit seven." She also explained to me that seven and a half is less than seven.

6. Relations between grade level and children's notations for and understandings of fractions

There are some interesting relations between the children's grade level and the types of notations and understandings of fractions they exhibited. Table II shows some distributions for children's answers in three categories.

6.1. *Halves are little bits*

A greater percentage of children in first grade interpreted fractions or halves as "little bits." At first glance, this might be surprising, given the fact that these children have begun to receive direct instruction regarding fractions and how to make notations for them. However, it may be that this instruction is leading children to certain misinterpretations regarding fractions, such as this one. Among the first grade children, the idea that "half" is something smaller than one, permeates their understanding of all numbers connected to half. If half is something smaller, then seven and a half must be smaller than seven. We see here how children assimilate this apparently school-transmitted fact with their previous understandings of number, trying to make sense of it all in a coherent way.

6.2. *Fractions' magnitude*

A greater percentage of Kindergarten students were able to explain the different magnitudes of fractions in relation to each other. For example, that seven and a half is more than seven, and that eight and a half is more than eight. This observation should probably be taken together with the previous one. Perhaps, because first graders are conceiving of fractions as little bits, it is difficult for them to understand their relative magnitude.

6.3. *Conventional notations for fractions*

Still, in spite of these two previous points, a greater percentage of first grade students were able to provide conventional notations for fractions.

TABLE II
Distributions of children's answers relative to their grade level

Types of notations or meanings	Halves are little bits	Can order fractions in terms of their magnitude	Provide a conventional notation for fractions
Kindergarten	14% (1/7)	86% (6/7)	14% (1/7)
First grade	64% (9/14)	29% (4/14)	38% (5/13)

7. REFLECTIONS

The data here reported illustrate that, as indicated by Pothier and Sawada (1983) and Empson (1999), young five and six-year-old children develop ideas about fractional numbers, showing that they are thinking and reflecting about this aspect of number and this area of mathematics. Unlike previous research, however, the research here reported explored these issues *through the lens of children's notations*. Thus, the novel perspective is that these young children exhibit the ideas previously described in the literature involving children the same age and older (e.g., Ball, 1993; Empson, 1999; Kamii, 2000; Mack, 1990; Pothier and Sawada, 1983; Sáenz-Ludlow, 1994, 1995) *through their notations*, and not just conceptually, devoid of any representational context or tool (see Ball (1993) for a description of representational contexts in the area of fractional numbers, and Empson (1999) for a discussion of the role of representational tools).

The data therefore indicate that it would be important to integrate notations into our study of young children's understanding and learning of fractional numbers. Their productions of spontaneous notations provide windows to develop more nuanced and complete pictures of their ideas about fractions. Sáenz-Ludlow (1995), for example, has proposed that, "initially, it could be necessary to accept children's informal written representations as transitional notations before conventional notations are negotiated with them" (p. 101), and that "children's use of natural language to verbalize their mental activity could serve as a first step toward symbolization" (p. 102). Thus, there is an admission that it is important to integrate both explorations: conceptual and notational. Further, as Empson (1999) states, the use of representational tools allows for the creation of "new forms of thinking", and not just "means of representing ideas" (p. 289). However, Mack (1990) states that studies she had reviewed had not, "addressed questions related to the ways that students can build on informal knowledge to give meaning to fraction symbols and procedures" (p. 17).

An exploration of children's notations for and understandings of fractions reveals the ways in which they create a logic of their own in trying to make sense and in gradually appropriating the conventional world of fractions, in this case. The process that children traverse in developing conventional notations for and understandings of fractions is exactly that: a (gradual) process. It is a constructive process, in which the relations between conceptual and notational understandings are complex. Children are exposed to conventional notations for fractions from a very early age. They are immersed in a world that uses notations for fractions and fractions in everyday life. They assimilate those aspects that are meaningful or salient for them in some way and incorporate them into their understandings about fractions in idiosyncratic ways.

In addition, fractions permeate multiple aspects of everyday and mathematical contexts (as we saw in children's differing understandings of fractions for sharing and partitioning and for measurement of time). Each of these contexts highlights particular aspects of fractional numbers. These different contexts need to be explored with children early on, in order to help them develop more robust understandings about fractions. An understanding of all the complexity of fractional number must include an understanding of the ways in which that complexity can be expressed. An over-emphasis on the ideas of fractions as they relate to sharing and partitioning does not allow for a full exploration of all the implications of the notational aspects of fractional number.

The research here reported also indicates that there is a dire need for future research that explores in more systematic ways the process through which both notations and conceptualizations about fractions are developed, over time. In addition, the role of different representational contexts (Ball, 1993) and tools (Empson, 1999) needs to be considered and explored: the ways in which different contexts highlight different aspects of fractions, and the impact of these contexts on the development of fractional understanding. Further, this paper explores young children's spontaneous notations for fractions; future research should also explore young children's interpretations of conventional notations for fractions.

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NOTES

1. Notice how different Nick's fives are in the different notations (see Figures 2 and 3), from one interview to the next. While his first notation for five (see Figure 2) is conventional, his second one (see Figure 3), from a few weeks later, reflects some trouble in making

the correct shape. As I have argued elsewhere (Brizuela, 2004), we cannot use his graphical performance on the task to evaluate his conceptual understanding – there is no obvious or simple one-to-one correspondence or match between both. Also, the fact that his notation is somewhat "worse" from one week to the next does not necessarily indicate that his conceptual understanding of fractions is worse.

2. Another interesting connection worth exploring is the use of "dummy numbers", such as 0, to stand for "half". Alvarado (2002) and Brizuela (2001, 2004) have observed similar behaviors in young children making notations for the digits in the tens place in two-digit numbers. As a dummy number, the zero is a notation that stands in for parts of numbers that children know should be shown but that they are unsure of how to do so. Quinteros (1997) has also reported on the use of "dummy letters" by young children. According to Quinteros, when children use dummy letters, "they are unsure of having used the adequate letter, but use them anyway and these letters come to function as dummy letters. These dummies. . .are included to substitute for a letter they are sure should be included in the written word, without knowing which it is" (p. 39).

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