

# Second-Best Optimal Rules for Excludable Public Goods

A thesis  
submitted  
by  
Jongsang Park

In partial fulfillment of the requirements  
for the degree of

Master of Arts  
in  
Economics

TUFTS UNIVERSITY

May 2006

Adviser: Gilbert E. Metcalf

## **Abstract**

While we define public goods only as non-rival, one can find many public goods that are excludable in reality. For those excludable public goods, the public provision agency could prevent some individuals from consuming the public good and charge other individuals to use it in a second-best economy. I first model excludable public goods, which provides a generalized public good model. With this model, I study the conditions where a positive price for an excludable public good is desirable in the framework of a nonlinear income tax model using a self-selection approach. I also derive an optimal provision rule for excludable public goods, which is a modified version of the famous Samuelson Condition. Finally, I study sufficient conditions where the second-best optimal level is higher (and lower) than that of the first-best level.

## **Acknowledgement**

First and foremost, I would like to thank my advisor, Professor Gilbert Metcalf, for all his guidance and support over the past year. When I was working with him as a research assistant, he encouraged me to further develop my initial work with him into my master's thesis. He is also the one who inspired me to work on a theoretical paper which I had not thought about for my master's thesis. Without his help and support, I would not have written this thesis. I would also like to thank my readers on the thesis committee, Professor Sharun Mukand and Professor John Straub, for their help and invaluable comments throughout the writing process as well as during the defense. I would also like to recognize Professor Yannis Ioannides, who supported and directed me towards economic research during my time at Tufts. I am also grateful to my family and friends for all of their support and love. Finally, I would like to express my deep gratitude to my fiancée, Soryun, whose encouragement and love made it possible for me to write this thesis.

# Table of Contents

<b>Abstract</b>	<b>ii</b>
<b>Acknowledgement</b>	<b>iii</b>
<b>I. Introduction</b>	<b>1</b>
<b>II. Literature Review</b>	<b>4</b>
1. Second-Best Public Good Provision	5
2. Excludable Public Goods	9
<b>III. A New Approach to Modeling Excludable Public Goods</b>	<b>11</b>
1. Modeling Excludability	11
2. Reclassification of Public Goods	17
<b>IV. Optimal Rules for Excludable Public Goods</b>	<b>21</b>
1. The Model	21
2. Desirability and Optimality of Prices for Excludable Public Goods	26
3. Optimal Provision Rule for Excludable Public Goods	30
<b>V. Optimal Provision Level Comparison</b>	<b>34</b>
1. A Sufficient Condition for Underprovision	38
2. A Sufficient Condition for Overprovision	40
3. Effect of Excludability on Public Good Provision Level	42
<b>VI. Conclusion</b>	<b>44</b>
<b>Appendices</b>	<b>46</b>
<b>References</b>	<b>58</b>

## I. Introduction

A public good, as opposed to a private good, is traditionally characterized as non-excludable and non-rival. A good is considered non-excludable if it is impossible for the planner to prevent users from consuming it, while a good is non-rival if one individual's consumption does not affect another individual's consumption opportunities. Some classic examples of public goods are lighthouses, national defense, fresh air and public parks. However, out of the above mentioned two attributes, non-excludability can be arguably omitted. Rosen (1999) defines a pure public good as follows:

*“A pure public good is nonrival in consumption. This means that once the good is provided, the additional resource cost of another person consuming the good is zero ... In contrast, a private good is rival in consumption.”*

He then treats non-excludability only as a possible attribute of a public good.<sup>1</sup> In this thesis, I will follow the definition of Rosen and describe a public good only as non-rival; hence it is possible for some public goods to be non-rival and excludable. These public goods are referred to as excludable public goods, and I will explore questions associated with these public goods.<sup>2</sup>

One clear example of an excludable public good is an uncongested toll bridge built by a local government. Once it is constructed, the marginal cost for an individual to use the bridge is almost zero. Since an individual's usage of the bridge does not hurt another individual's opportunity of using it, this toll bridge fits well in the category of a public good. However, the government can install a toll booth to charge a positive toll for

---

<sup>1</sup> See p. 62 and p. 81, Rosen (1999). Rosen argues that many (not all) public goods are also non-excludable.

<sup>2</sup> Note that the excludable public goods considered here are different from club goods. Although club goods are also public goods that can be excludable, analysis on club goods must be accompanied by the assumption of congestion when consumed by many consumers. However, in this research, I rule out congestion: even without congestion, one can easily find situations where the planner wants to charge user fees because of efficiency issues or redistributive issues.

users to allow them to decide whether or not to use the bridge, or how many times, if at all. That is, the government can exclude some individuals from using it.

As seen in the example above, the planner (i.e. the government) could charge individuals for consuming an excludable public good, or preventing consumption of an excludable public good. Therefore, the government has one more option to finance the provision of these goods other than by collecting a tax: that is, instituting a price system.

Note that the marginal cost of accommodating another consumer is zero because consumption is non-rival for any public good. Consequently, the price charged by the planner should be zero in the absence of distortions in the ideal first-best economy.

However, a second-best economy necessarily involves some kinds of distortion, which makes a positive price optimal for the economy. Thus, this second-best alternative for financing public goods available for a government naturally raises the following questions. Under what conditions is it desirable to implement a price system? Under those conditions, what would be the optimal price? Will the optimal size of the public good financed through a price system be bigger or smaller than the first-best optimal size?

In this thesis, I provide a generalized public good model which can encompass various cases of excludable and non-excludable public goods within it. Then I show that, when the government considers redistribution within the society, a positive price for an excludable public good is desirable if the public good consumption level increases with the amount of leisure. This result is analogous to the classic result in the literature, such as Corlett and Hague (1953-1954), studying the relationships between direct taxation and indirect taxation: commodities that are complementary to leisure should be taxed. I then derive the optimal pricing formula for excludable public goods by modifying the

Samuelson Condition with additional consideration for self-selection effect and revenue effect. I also show that we may well have overprovision of excludable public goods in a second-best economy, relative to the first-best economy, under the conditions which would imply underprovision if the public goods were non-excludable.

The remainder of this thesis is organized as follows. I first model the excludability of public goods in a formal way. In the next section, I employ a nonlinear optimal income tax structure with a self-selection approach to study optimal rules for excludable public goods. In the last section, I compare the optimal level of public good provision of a second-best economy with that of the first-best economy.

## II. Literature Review

This thesis deals with analyses of excludable public goods financed by second-best taxation. Hence, I will review the literature in two following areas: 1) second-best public good provision research; and 2) recent excludable public good research. To begin with, I discuss the famous Samuelson Condition for the optimal provision rule for public goods because it is a benchmark finding in public good research. In Samuelson (1954) and (1955), the Samuelson Condition indicates that a planner should provide a public good to the point where the sum of all individuals' marginal benefit of consumption meets the marginal cost of provision. Thus, the formula of the optimal provision rule for pure public goods is:

$$(L-1) \quad \sum MRS = MRT$$

Interpreting this condition intuitively,  $MRS$  can be considered as the marginal willingness to pay in dollar terms, and  $MRT$  as the marginal direct cost also in dollar terms. Since a public good is one that all individuals in an economy consume together, the optimal level of public good provision should be where the sum of the money that all consumers are willing to pay is equal to the marginal cost of provision.<sup>3</sup> Note that the derivation of this condition assumes that tax revenues for financing a public good come from lump-sum taxes, which describes the first-best economy.

---

<sup>3</sup> A consumer's optimal decision rule for a *private* good is that individual's  $MRS$  is equal to  $MRT$ , meaning that the consumer's marginal willingness to pay should be equal to the marginal cost, in the absence of other distortions.

## 1. Second-Best Public Good Provision

Before the Samuelson Condition was developed, Pigou (1947) recognized the importance of distortion involved in a second-best taxation, and argued that the “indirect damage” would add costs to direct social costs financing public goods, and that the optimal level of public good provision financed by distortionary taxes should be less than the optimal level financed by lump-sum tax. However, the subsequent literature studying the relationships between public good provision and second-best distortionary taxes reported that the conjectures in Pigou are not correct. That is, distortionary taxes of a second-best economy might actually reduce total social costs under certain condition. Furthermore, an optimal provision rule cannot tell us about the size of the public good. For describing a second-best economy, the literature on this topic employs two different taxation systems to finance public good provision: optimal commodity taxation and optimal nonlinear income taxation.

### 1.1) Public Goods Financed by Optimal Commodity Taxation

One line of research of this literature uses an optimal commodity tax. Atkinson and Stern (1974) clearly showed that the distortionary effect of commodity taxes could decrease total social cost. They found that the optimal provision rule with distortionary commodity taxes requires additional effects to the Samuelson Condition:<sup>4</sup>

$$(L-2) \quad MRT = \frac{\alpha}{\lambda} \sum MRS, \quad \text{where } \frac{\alpha}{\lambda} = 1 - \sum t_i \frac{\partial X_i}{\partial I} + \sum t_i \frac{S_{ik}}{X_i}.$$

In order to see how this optimal rule is different from the Samuelson Condition, the term

---

<sup>4</sup> The original equation in Atkinson and Stern is that  $MRT = \frac{\alpha}{\lambda} \sum MRS + \frac{\partial}{\partial e} [\sum t_i X_i]$ . However, the second term on the right hand side was assumed to be zero by them, because it was not “central to their paper.”

$\frac{\alpha}{\lambda}$  should be compared with unity by considering two additional terms on the right hand side of the second equation:  $\sum t_i \frac{\partial X_i}{\partial I}$  and  $\sum t_i \frac{S_{ik}}{X_i}$ . First, the term  $\sum t_i \frac{S_{ik}}{X_i}$  represents the negative distortionary effect resulting from deadweight loss of commodity taxation, as Pigou argued (1947). This negative effect would make the term  $\alpha$  less than  $\lambda$ , so that  $\sum MRS > MRT$ . Secondly, the other term  $\sum t_i \frac{\partial X_i}{\partial I}$  represents the revenue effect, or the marginal tax revenue of income, and is one of the effects that Pigou overlooked. This term could be negative or positive, depending on the normality of the taxed good. If the taxed good is labor (or if the subsidized good is leisure), the sign of the term depends on whether leisure is inferior or normal. Thus, this term may work in the direction of  $\alpha > \lambda$ , which would make  $\sum MRS < MRT$ . Considering these two effects together, Pigou's conjecture turned out to be incorrect. Also, Atkinson and Stern showed that a modified version of the Samuelson Condition with these two additional effects does not necessarily describe the optimal size of the public good, as opposed to conventional arguments that a bigger  $MRS$  would indicate a smaller provision level. Although their research is important for second-best public good research in the sense that it was the first to verify clearly how a second-best optimal provision rule deviates from the first-best rule, their analysis was restricted to the case of identical individuals so that redistribution issues were ignored. Later, King (1986) extended their result to account for individual heterogeneity and derived a more generalized version of the Samuelson Condition with an additional effect along with the effects found by Atkinson and Stern. The new effect describes redistributive concerns, implying that when an individual with a lower social

marginal value of income (i.e. high ability-type individual) prefers the public good more than an individual with a higher social marginal value of income (i.e. low ability-type individual) does, increased provision of the public good will add a redistributive cost to the total social cost.

## **1.2) Public Goods Financed by Nonlinear Income Taxation**

The other line of public good research uses a nonlinear income tax. Since this thesis focuses on this tax system with a self-selection approach, I will first present the model of nonlinear income taxation of Stiglitz (1982), and then review the subsequent literature on public goods adopting the framework of the income tax model.<sup>5</sup>

Nonlinear income tax literature has assumed that the government has imperfect information on individuals' productivity and cannot observe the hours of labor supply. Thus, it imposes income tax based on individual's income itself, allowing for individuals' self-selection. Stiglitz first investigated nonlinear income tax issues in a two-type Pareto-optimizing economy where the government sets two self-selection constraints to prevent any type of individual from mimicking the other type of individual. When we assume that a high-ability person's indifference curve on the after-tax-before-tax plane is flatter than that of a low-ability person, only one of the two self-selection constraints is binding: this assumption is the agent monotonicity condition. According to Stiglitz, the "normal" case is where a high-type individual would mimic a low-type individual, an argument which the subsequent literature has followed. In this "normal" case, it should be noted that the marginal tax rate of a low-ability individual is positive, while the marginal tax rate of a

---

<sup>5</sup> The reason why the nonlinear income tax model is employed here is that, first, one can observe a nonlinear income tax system more easily than a linear tax system in a real world; and secondly, it can be useful to compare this research with recent research in similar frameworks, such as Boadway and Keen (1993), Edwards et al. (1994), Blomquist and Christiansen (2005), and Gaube (2005).

high-ability individual is zero, and that social welfare in a second-best economy is strictly less than that of the first-best economy. Later, Christiansen (1984) made close links among the taxation literature that had explored the relationships between income taxes and commodity taxes, and confirmed that leisure-complementarities should be taxed within the existence of optimal income taxation.

Boadway and Keen (1993) were the first that questioned the optimal public good provision in the framework of this nonlinear income tax, and developed a modified Samuelson Condition accordingly. Their modified condition for optimal provision involves an additional effect on social cost associated with mimicking incentives due to self-selection. This effect could well be negative or positive, since this effect may add (or reduce) social cost when increased provision of public goods gives a mimicker more (or less) incentives for mimicking, which the society wants to prevent. Thus, their finding is another verification of how Pigou's conjecture is incorrect. Edwards et al. (1994) later studied a similar public good provision question with nonlinear income taxes supplemented by a commodity tax system, and found out another effect on the social cost due to the additional tax revenue gathered from commodity taxes. Accordingly, their optimal provision rule involves one more effect in addition to the mimicking effect reported by Boadway and Keen, namely a revenue effect. This revenue effect results from compensated changes in taxed goods demand due to increased public good provision.

It is important to note that a price for excludable public goods resembles a commodity tax for the following two reasons. First, a positive price for public goods creates the same distortion as that created by a commodity tax, because consumers are

being charged more than the marginal cost.<sup>6</sup> Second, positive prices are just another way for the government to finance its expenditure. Therefore, one can expect that this thesis on a price system for public goods with a framework of optimal income taxation is on the same track as Christiansen (1984) and Edwards et al. (1994) who explored optimal taxation rules and public good provision rules when commodity taxes supplement income taxes.

Recently, Gaube (2005) studied sufficient conditions under which the optimal provision level of public goods financed by a nonlinear income tax deviates from that of the first-best economy. He shows that when  $\sum MRS > MRT$ , a sufficient condition for underprovision requires two more assumptions: the public good is normal as well as Hicksian substitute for leisure. That means, without these assumptions,  $\sum MRS > MRT$  cannot guarantee underprovision of the public good. Thus, Gaube's study responds to the second part of Atkinson and Stern (1974) with linear income tax, demonstrating that an optimal provision rule does not indicate the optimal provision level.

## 2. Excludable Public Goods

Relative to non-excludable public good research, there has been much less literature on the second-best provision rules for excludable public goods. Some previous research including Oakland (1974), Burns and Walsh (1981), and Fraser (1996) studied topics related to excludable public goods. Oakland (1974) showed that if exclusion is costless, public good provision would be suboptimal. Burns and Walsh (1981)

---

<sup>6</sup> Recall that the marginal cost for additional consumption of public goods is zero.

investigated various pricing schemes for excludable public goods. However, their studies are not relevant to the line of inquiries here, because they focused on market private provision of excludable public goods. Fraser (1996) showed that public goods would be underprovided with self-choice relative to the mandatory subscription, but he did not consider distortionary income taxation, which is essential for the analysis in this thesis.

Recently, Blomquist and Christiansen (2005) studied the price question of public provision of excludable public goods with a nonlinear income tax and a self-selection approach. They argued that only when marginal valuation of the public good increases in leisure may a positive price be desirable. Furthermore, they argued that even under this condition, a price smaller than the optimum might not improve the welfare, and may even decrease welfare. In other words, if the planner could not be sure to set the price at the optimal level, it might be preferable not to charge any price at all: thus, implementation of a price system would be risky. However, these results crucially hinge on Blomquist and Christiansen's own modeling of individual public good consumption, which will be shown in Chapter III to be too restrictive to be applicable to many public good cases.

### III. A New Approach to Modeling Excludable Public Goods <sup>7</sup>

In this chapter, a new approach for modeling public good consumption will be developed. I will first model excludability of public goods, and then discuss how this model is applicable to various public good models including non-excludable cases.

#### 1. Modeling Excludability

In order to model excludability of public goods in a formal way, it is necessary to recognize that excludable public good consumption levels are varied among individuals, as opposed to non-excludable public good consumption. To see this variation, let us begin by considering an example of excludable public goods. In the toll bridge example mentioned earlier in Chapter I, the government can exclude some users from consuming the public good (i.e. the bridge) and, in the meantime, let other users decide how many times to use the bridge. This makes, as a consequence, the public good consumption across users varying, due to the individual heterogeneity with respect to preferences or income levels. Keeping in mind that excludable public good consumption varies, let  $g^i$  be the  $i^{\text{th}}$  individual's public good consumption and  $G$  be the whole amount of public goods provision. Note that, in a non-excludable public good case, public good consumption for every individual is set to be  $G$ , so the literature on non-excludable public goods has included only  $G$  as the argument in the utility function. However,  $g^i$  should be used here as excludable public good consumption of the  $i^{\text{th}}$  individual, meaning

---

<sup>7</sup> This chapter and the next chapter are based on the joint work of Gilbert Metcalf and the author.

that consumption of public good is varying across individuals. For the first step, it is reasonable to set  $g^i$  to be a function of  $G$ . As the government increases the quality of the toll bridge (i.e. higher  $G$ ), the public good consumption of any individual increases (i.e. higher  $g^i$ ) per each access. Therefore, an individual's consumption  $g^i$  is a strictly increasing function of  $G$ . The next step is to model differences in individual public good consumption by incorporating the level of access to the public good in this function of  $g^i$ . Differences in public good consumption across individuals result from the differences in their levels of access: in the example of the bridge again, every time you access to the bridge, you pay for a one-time-pass at the toll booth. I denote the level of access by  $v^i$ .<sup>8</sup> Thus, the  $i^{\text{th}}$  individual's public good consumption  $g^i$  can be written as  $g^i(v^i, G)$ . For the function of  $g_i$  to be meaningful, the following conditions are imposed on  $g^i$ :

$$(1-1) \quad \frac{\partial g^i}{\partial G} > 0$$

$$(1-2) \quad \frac{\partial g^i}{\partial v^i} > 0$$

$$(1-3) \quad \frac{\partial^2 g^i}{\partial G \partial v^i} > 0$$

The first two conditions (1-1) and (1-2) are quite obvious requirements indicating that  $g^i$  is an increasing function of both  $G$  and  $v^i$ , while the condition (1-3) means that the marginal public good consumption of access increases with  $G$ . The final assumption of

---

<sup>8</sup> I briefly discuss how the level of access to public goods,  $v^i$ , should be interpreted in the context of other examples of public goods. In the case of a public park with an entrance charge per visit,  $v^i$  is interpreted as the number of visits; in the case of public TV with an hourly fee,  $v^i$  is interpreted as hours of watching TV.

the  $g^i$  function is that the marginal consumption of access is independent of access itself.

That is,

$$(1-4) \quad \frac{\partial g^i}{\partial v^i} = h(G).$$

By solving the differential equation of (1-4), I derive the general function of  $g^i$

satisfying all the conditions (1-1) to (1-4):<sup>9</sup>

$$(1-5) \quad g^i = v^i \cdot h(G) \quad , \text{ where } h(G) > 0 \text{ and } h'(G) > 0 .$$

Hence, the simplest functional form of  $g^i$  satisfying all the above assumptions might be

$$g^i = v^i \cdot G .$$

Now, let us consider how an individual optimizes his utility with excludable public goods. Suppose the  $i^{\text{th}}$  individual enjoys leisure  $L^i$ , a private good  $X^i$ , and an excludable public good  $G$  with the level of access  $v^i$ . Prices for  $X$  and  $v$  are  $p_x$  and  $p_v$ , respectively. Also, let  $Z$  be the time endowment and  $w^i$  be the wage rate. Then, the individual optimization problem is:<sup>10</sup>

$$(2-1) \quad \begin{aligned} & \max U^i (X^i, g^i, L^i) \\ & \text{subject to } p_x \cdot X^i + p_v \cdot v^i + w^i \cdot L^i = w^i \cdot Z \end{aligned}$$

This optimization problem might be found questionable for the following two reasons.

First, this problem looks a bit unusual when one applies general microeconomic analyses, such as Roy's Identity or the Slutsky decomposition, because we have different variables

---

<sup>9</sup> Although public good consumption is assumed to be linear in the level of access (visits), it does not imply that utility should also be linear in the level of access, because utility does not have to be linear in public good consumption.

<sup>10</sup> For simplicity, I here ignore income tax and commodity tax to see how the modeling of excludability can be applicable to consumer problems *in general*. I will work with income taxation from the next chapter on.

( $g^i$  and  $v^i$ ) in the utility function on the one hand, and in the budget constraint on the other. Second, demand for  $v^i$  might go to infinity if the government sets  $p_v$  at zero. Thus I need to address these two issues first.

For the first issue, I can easily convert this problem into a familiar one by defining a price for the excludable public good itself,  $p_g = \frac{p_v}{h(G)}$ . Then,  $p_g$  is the price for the public good consumption, while  $p_v$  is the price for the access to the public good. It is reasonable that  $p_g$  is increasing with  $p_v$  and decreasing with  $h(G)$  or  $G$  - in the bridge example again, the price of *consuming* the bridge will decrease when the toll gets cheaper (thus,  $p_v$  decreases) or when the quality of the bridge improves (thus,  $G$  increases), with other things being equal. Using this new price  $p_g$ , expenditure on public good consumptions can be expressed as  $p_g \cdot g^i$  ( $= p_v \cdot v^i$ ). Then the problem becomes (2-1) familiar:

$$(2-2) \quad \begin{aligned} & \max U^i(X^i, g^i, L^i) \\ & \text{subject to } p_x \cdot X^i + p_g \cdot g^i + w^i \cdot L^i = w^i \cdot Z. \end{aligned}$$

Now we have three variables,  $X^i$ ,  $g^i$  and  $L^i$ , in both the objective function and the budget constraint.

The second issue is more subtle. Although one could easily avoid the risk of having infinite demand by simply assuming that the demand is not limitless, I would solve this problem in a formal way by introducing a positive private cost to the consumption of the public good. Since consuming an excludable public good would involve time, a private cost denoted by  $p_c$  ( $> 0$ ) can be added to the nominal price for

public good,  $p_v$ . Then the problem (2-1) becomes (2-3) and I do not have infinity demand for  $v$  with zero  $p_v$ :

$$(2-3) \quad \begin{aligned} & \max U^i (X^i, g^i, L^i) \\ & \text{subject to } p_x \cdot X^i + (p_v + p_c) \cdot v^i + w^i \cdot L^i = w^i \cdot Z. \end{aligned}$$

Note that the amount of leisure  $L^i$  here (2-3) is less than that of (2-1), because this private cost is assumed to be associated with consumption of time. With the same logic, I can also modify problem (2-2) with the introduction of a private cost to get:

$$(2-4) \quad \begin{aligned} & \max U^i (X^i, g^i, L^i) \\ & \text{subject to } p_x \cdot X^i + \left( p_g + \frac{p_c}{h(G)} \right) \cdot g^i + w^i \cdot L^i = w^i \cdot Z \end{aligned}$$

Having modeled excludable public goods, I turn to brief discussion of the applicability of the excludable public good model developed by Blomquist and Christiansen (2005) to a real world. In Blomquist and Christiansen (2005), the government provides  $G$  units of an excludable public good to which individuals can obtain access by paying a per unit price  $q$ . Upon payment of  $q \cdot g$ , individuals enjoy  $g$  units of the public good with the capacity constraint that  $g \leq G$ . Examples they gave were an art gallery with  $g$  measuring the number of rooms visited and  $G$  the total number of rooms available; and an uncongested toll road with  $g$  measuring the number of sub toll roads that a driver uses and  $G$  the total length of the road available. Thus, the capacity constraint that  $g \leq G$  plays a key role in their model: in the example of the art gallery, unless a guest visits all rooms available (or unless a guest is “rationed”), increased  $G$  does not affect the guest’s utility; in the example of the toll road, unless a driver uses up the entire toll road available, increased  $G$  does not affect the driver’s utility.

However, one can realize that they might have ignored the possibility that the government can increase  $G$  to make “unrationed” consumers as well as “rationed” consumers happier by improving the quality of public goods, for example, by displaying a better quality of arts in a single room, or by providing a better quality of road surface of the toll road: that is, they treated increased  $G$  only as expansion in terms of quantity.<sup>11</sup> Furthermore, they seem to assume implicitly that consumers can purchase the public good only once. For example, if a driver uses the entire road once a day and another driver uses it twice a day, both would be considered, in their model, to consume the same amount of the public good,  $G$ , because both are “rationed.” However, it is obvious that the second driver consumes this public good more than the first driver. Thus, their model of excludable public goods is seriously problematic in two aspects: ignorance of quality of public goods and assumption of one-time consumption. In fact, their examples can be viewed as bundles of multiple public goods with an all-or-nothing price scheme, and their model is better understood as a model of a bundle of multiple public goods where increased  $G$  means attaching one more good to the bundle. Of course, it may well be that their modeling of public goods is appropriate in certain cases, but it is not applicable to most public goods, so their results could also be misleading.

Now I am ready to apply the model (2-3) to excludable public good analyses in a second-best economy. I will use this model to explore indications of a price system within the frameworks of a nonlinear income tax system.

---

<sup>11</sup> Even a quantity expansion could make an unrationed visitor of an art gallery happier due to larger options of choice.

## 2. Reclassification of Public Goods

Before proceeding to the main inquiry of this thesis with this new model, I will discuss a surprising flexibility of the model for a variety of public good examples and the public good literature.<sup>12</sup> I will demonstrate here that the model developed in the last section is easily applicable to non-excludable cases as well as to excludable cases. To see how this is so, I need first to classify public goods in two aspects: possibility of exclusion and endogeneity of consumption level. Thus, I have four different types of public goods, shown in Table 1.

Table 1. Public Good Examples for Each Type

		Possibility of exclusion	
		Impossible	Possible
Endogeneity of consumption level	Exogenous	Case 1. national defense	Case 2. compulsory public education
	Endogenous	Case 3. fresh air	Case 4. uncongested bridges, public parks, or lighthouses

I will briefly explain the two criteria. The first criterion, possibility of exclusion, is one that questions whether the planner is able to exclude a consumer from a public good in reality. Thus, this criterion is related to recent technology developments. For example, a public TV service had been considered impossible for exclusion, but it has now become possible for a provider to technically exclude some consumers from watching it. The second criterion, endogeneity of consumption level, involves the question of whether a consumer can choose his own consumption level. For most public

---

<sup>12</sup> For the purpose of comparing this model with that of previous literature, let us suppose  $h(G) = G$ .

goods, consumption level can be endogenously chosen, while some public goods are consumed with consumption level being fixed exogenously. Thus, the first criterion, possibility of exclusion, is a matter of the *planner's* ability to exclude, while the second criterion, endogeneity of consumption level is related to a *consumer's* ability to choose. Now I turn to discussions of the application of this model to public good examples for each box of the table 1.

In the upper-left box (denoted by Case 1), the public good is non-excludable and consumption level is exogenous. This is how a public good has been traditionally treated and how research on public good has been done. The most suitable example for this case is national defense. A government cannot exclude any citizen from consuming national defense, and no citizen can choose his consumption level. With this new model, the access level  $v^i$  for every citizen is fixed, so we can normalize  $v^i$  as constant at unity and the government cannot charge a price. The private cost can also be dropped, since an individual does not pay any opportunity cost for consuming this kind of public goods.<sup>13</sup> Thus, problem (2-3) becomes:

$$\begin{aligned} & \max U^i (X^i, G, L^i) \\ & \text{subject to } p_x \cdot X^i + w^i \cdot L^i = w^i \cdot Z, \end{aligned}$$

which is just a traditional public good problem. In other words, one can just set  $v^i$  constant to make this model suitable for some non-excludable public goods of exogenous consumption level, such as national defense.

In the upper-right box (Case 2), the public good is excludable and consumption level is exogenous. That is, an individual cannot choose the consumption level, while the

---

<sup>13</sup> In terms of modeling aspect, there is no risk of infinity demand, so we do not need to implement a positive private cost.

planner is able to exclude some individuals from consuming it. Compulsory government-funded public education is a good example for this category. Once a twelve-year-old child consumes the compulsory education by going to the public school, he will consume the same amount of public education as other students. Of course, the planner is *able* to exclude some students from consuming the education. In order to fit this type of public good into the model, one can again normalize the access level as unity, which makes  $g^i = G$ . Then the individual problem (2-3) becomes:

$$\begin{aligned} & \max U^i (X^i, G, L^i) \\ & \text{subject to } p_x \cdot X^i + w^i \cdot L^i = w^i \cdot Z - (p_v + p_c) = w^i \cdot Z', \end{aligned}$$

which becomes a traditional public good problem as well.

The third case in the lower-left box now allows for varying levels of consumption of public goods. Nevertheless, the government cannot exclude any consumer from consuming this type of public goods, mainly due to technology restrictions. A good example is fresh air, which is also often cited as an example of public goods. If an individual is a very active person who also enjoys jogging everyday, while his friend is a sedentary person who hardly ever goes outside, then consumption levels of fresh air should be different between two individuals. Since the government cannot charge any price for consuming fresh air, the individual problem becomes:

$$\begin{aligned} & \max U^i (X^i, v^i \cdot G, L^i) \\ & \text{subject to } p_x \cdot X^i + p_c \cdot v^i + w^i \cdot L^i = w^i \cdot Z \end{aligned}$$

Now, if we interpret  $v^i$  as number of hours of enjoying fresh air outside, it can also be possible to treat  $v^i$  as part of consuming leisure. Then private cost,  $p_c$ , should be exactly the same as the wage rate because a private cost represents an opportunity cost. Let

$TL^i$  be total amount of leisure calculated by  $L^i + v^i$ . Then the budget constraint becomes  $p_x \cdot X^i + w^i \cdot TL^i = w^i \cdot Z$ . Therefore, the individual problem above also becomes a traditional problem of a non-excludable public good model by setting zero prices for the access to the public good.<sup>14</sup>

Finally, the lower-right box represents the case where excludable public good consumption is endogenous, the situation on which I mainly focus in this thesis. Thus, the example given in Chapter I, an uncongested toll bridge, falls into this category. A more classic example for this category is a lighthouse, first introduced in Coase (1974). Because ships would come near the lighthouses only if they planned to enter or leave British ports, it was possible to levy a usage charge. If ships did not pay this fee, they would be denied future entry into (or exit from) these ports: that is, they would be excluded from consumption of this public good. Thus, a lighthouse is really an excludable public good with endogenous consumption level. The individual problem for this kind of public goods should remain the same as (2-3).

So far, I have verified that the model for excludable public goods, which I developed in the last section, is well applicable to most public good examples no matter if the public goods are excludable or non-excludable. Thus, the public good model developed here is a generalized public good model. I will confirm again the flexibility of this model in Chapter IV, by showing how the optimal provision rule with this model can also encompass the optimal provision rule for a non-excludable public good model.

---

<sup>14</sup> Note that the differences in the public good argument, which is  $v_i \cdot G$ , in the utility function can be interpreted as the same amount of  $G$  with different marginal valuation of public good due to heterogeneous preference.

## IV. Optimal Rules for Excludable Public Goods

In this chapter, I apply the model developed so far to the nonlinear income tax model with a self-selection approach. I will study the desirability of a positive price and derive the optimal pricing rule and provision rule for excludable public goods. I will also discuss how the optimal rules for excludable public goods can be compared to the optimal rules for non-excludable public goods.

### 1. The Model

Consider an economy of two types of individuals with wage rates of  $w^1$  and  $w^2$  ( $w^2 > w^1$ ). Each individual consumes one private good and one excludable public good. Let  $c^i$  be private good consumption of a type- $i$  individual,  $H^i$  be hours of labor supply,  $L^i$  be hours of leisure, and  $g^i$  be excludable public good consumption, as defined in the last chapter. For simplicity, I assume  $g^i = v^i \cdot G$  by setting  $h(G) = G$  where  $v^i$  is the level of access to the public good. Treating the private good as the numeraire, the price for the private good is set to be one, and  $p_v$  is the non-negative price for the access to the public good. Utility function of this consumer is given as

$$(3-1) \quad U^i(c^i, g^i, H^i), \quad \text{for } i = 1, 2.$$

Within the framework of nonlinear income taxation and the self-selection approach of Stiglitz (1982), the government can observe only the individual's total income,  $Y^i = H^i \cdot w^i$ , not the amount of labor supply ( $H^i$ ), nor the wage rate ( $w^i$ ). Thus, it imposes income taxes solely based on the individual's income. That is, the nonlinear

income tax is a function of  $Y^i$ :  $T = T(Y^i) = T(H^i \cdot w^i)$ . The constraint that this consumer faces is:

$$(3-2) \quad c^i + p_v \cdot v^i = H^i \cdot w^i - T(Y^i) = B^i, \quad \text{where } B^i \text{ is disposable income.}$$

As discussed in the last chapter, implementation of a private cost is needed to avoid infinite demand for  $v$  when  $p_v$  goes to zero. Let us suppose that one public good access requires  $e$  amount of time. Then, the individual time constraint will be:

$$(3-3) \quad Z = H^i + L^i + e \cdot v^i, \quad \text{where } Z \text{ is the time endowment.}$$

Since this individual will spend all the disposable income on private good consumption when the price for the public good is zero, the individual budget constraint in terms of disposable income is adequately given by (3-2). Combining the time constraint (3-3) and the budget constraint (3-2), the individual budget constraint in terms of total income is:

$$(3-4) \quad c^i + (p_v + e \cdot w^i) \cdot v^i + w^i \cdot L^i + T(Y^i) = w^i \cdot Z$$

Here, the private cost discussed in the last chapter is now being specified as  $e \cdot w^i$ . By setting  $p_v + e \cdot w^i = \tilde{p}_v$ , we get

$$(3-5) \quad c^i + \tilde{p}_v \cdot v^i + w^i \cdot L^i + T(Y^i) = w^i \cdot Z$$

Therefore, one can use either of the two budget constraints, (3-2) or (3-5), but in terms of disposable income  $B^i$ , the budget constraint (3-2) with  $p_v$  should be used. Note that infinite demand for  $v^i$  can still be prevented even with the budget constraint of (3-2). It will also be useful to define the price for public good consumption  $p_g$  for further

analysis, by defining  $p_g = \frac{p_v}{G}$ . Then, the public good expenditure is  $p_v \cdot v^i = p_g \cdot g^i$ .

Therefore, a type-i individual's optimization problem can be expressed as:

$$(3-6) \quad \begin{aligned} & \max U^i (c^i, g^i, H^i) \\ & \text{subject to } c^i + p_g \cdot g^i = H^i \cdot w^i - T(Y^i) = B^i \end{aligned}$$

Now, I am ready to derive indirect utility function for a type-i individual,  $V^i(B^i, Y^i, p_g, G; w^i)$ , by solving the maximization problem (3-6). As Edwards et al. (1994) suggested, one might think of this individual optimization problem as a two-stage problem: first, this individual takes a bundle  $(B^i, Y^i)$  offered by the government and decides how much labor to supply;<sup>15</sup> then, allocates  $B^i$  over private good consumption and public good consumption. Since  $w^i$  is a unique characteristic of a type-i individual, and is already specified with the superscription of the utility function, I can drop  $w^i$  out and write the indirect utility function as  $V^i(B^i, Y^i, p_g, G)$ .

Also, I may have another expression of the indirect utility function  $V^{*i}$  with  $p_v$  which will be used for the optimal provision level question in the final chapter.

$$(3-7) \quad V^i(B^i, Y^i, p_g, G) = V^i(B^i, Y^i, \frac{p_v}{G}, G) = V^{*i}(B^i, Y^i, p_v, G)$$

Note that both  $V^i$  and  $V^{*i}$  are all expressed in terms of the instruments that the government can control.

Turning to the government's problem, let the population size of each type of individual be  $N^1$  and  $N^2$ , respectively. The government wants to make Pareto-optimal policies with both income tax and prices for the access to the public good for financing provision of the public good, by maximizing the objective function  $V^1$  subject to a

---

<sup>15</sup> Note that the government must specify, in a taxation bundle, at least two variables out of a total of three: income  $Y^i$ , disposable income  $B^i$  and tax  $T$ . That means, a bundle  $(B^i, Y^i)$  offered by the government can instead be  $(Y^i, T(Y^i))$ . However, only  $T$  does not provide enough information on the labor supply and the disposable income, because the tax function  $T(Y^i)$  is not one-to-one with respect to  $Y^i$  in general.

minimum level of  $V^2$  with two more constraints, the budget constraint and the self-selection constraint as following:

$$(3-8) \quad N^2(Y^2 - B^2) + N^1(Y^1 - B^1) + p_g(N^1 \cdot g^1 + N^2 \cdot g^2) - m \cdot G \geq 0$$

$$(3-9) \quad V^2(B^2, Y^2, p_g, G) \geq V^2(B^1, Y^1, p_g, G)$$

Self-selection constraints are generally imposed to make a type- $i$  individual prefer the bundle of  $(B^i, Y^i)$  intended for him. In other words, the government wants to prevent mimicking behavior by setting self-selection constraints. Here, as in the most nonlinear income tax literature with self-selection, I focus on the situation where a high-ability individual would pretend to be a low-ability individual. As argued in Stiglitz (1982), Edwards et al. (1994), and Gaube (2005), this is the “normal” case, and the assumption of binding (3-9) implicitly assumes redistribution from type-2 individuals to type-1 individuals (i.e. rich households to poor households).<sup>16</sup> From now on, for brevity, I will express functions of a type-2 individual mimicking a type-1 individual by using “hat,” when I omit the arguments of functions for mimickers (i.e.  $\hat{V}^2 = V^2(B^1, Y^1, p_g, G)$ ).

With these ingredients, I formulize the government’s optimization problem:

$$(3-10) \quad \text{Max}_{B^1, Y^1, B^2, Y^2, p_g, G} V^1(B^1, Y^1, p_g, G) \quad \text{subject to}$$

$$(3-11) \quad V^2(B^2, Y^2, p_g, G) \geq \bar{V}^2$$

$$(3-12) \quad V^2(B^2, Y^2, p_g, G) \geq V^2(B^1, Y^1, p_g, G)$$

$$(3-13) \quad N^2(Y^2 - B^2) + N^1(Y^1 - B^1) + p_g(N^1 \cdot g^1 + N^2 \cdot g^2) - m \cdot G \geq 0$$

---

<sup>16</sup> The other self-selection case is that a low ability individual would mimic a high ability individual, which has not been emphasized in the literature. Also, as mentioned in Chapter II, the agent monotonicity condition can preclude the case where both of the self-selection constraints bind at the same time. See p. 8 more on this.

Note that the constraint (3-12) describes the “normal” self-selection constraint, and the constraint (3-13) is the budget constraint. The Lagrangian problem is:<sup>17</sup>

$$(3-14) \quad \Lambda = V^1(B^1, Y^1, p_g, G) + \beta[V^2(B^2, Y^2, p_g, G) - \bar{V}^2] + \rho[V^2(B^2, Y^2, p_g, G) - V^2(B^1, Y^1, p_g, G)] + \mu[N^2(Y^2 - B^2) + N^1(Y^1 - B^1) + p_g(N^1 \cdot g^1 + N^2 \cdot g^2) - m \cdot G]$$

The first order conditions are:

$$(3-16) \quad \frac{\partial \Lambda}{\partial B^1} = \frac{\partial V^1}{\partial B^1} - \rho \cdot \frac{\partial \hat{V}^2}{\partial B^1} - \mu \cdot N^1 + \mu \cdot p_g \cdot N^1 \cdot \frac{\partial g^1}{\partial B^1} = 0$$

$$(3-17) \quad \frac{\partial \Lambda}{\partial Y^1} = \frac{\partial V^1}{\partial Y^1} - \rho \cdot \frac{\partial \hat{V}^2}{\partial Y^1} + \mu \cdot N^1 + \mu \cdot p_g \cdot N^1 \cdot \frac{\partial g^1}{\partial Y^1} = 0$$

$$(3-18) \quad \frac{\partial \Lambda}{\partial B^2} = \beta \cdot \frac{\partial V^2}{\partial B^2} + \rho \cdot \frac{\partial V^2}{\partial B^2} - \mu \cdot N^2 + \mu \cdot p_g \cdot N^2 \cdot \frac{\partial g^2}{\partial B^2} = 0$$

$$(3-19) \quad \frac{\partial \Lambda}{\partial Y^2} = \beta \cdot \frac{\partial V^2}{\partial Y^2} + \rho \cdot \frac{\partial V^2}{\partial Y^2} + \mu \cdot N^2 + \mu \cdot p_g \cdot N^2 \cdot \frac{\partial g^2}{\partial Y^2} = 0$$

$$(3-20) \quad \frac{\partial \Lambda}{\partial p_g} = \frac{\partial V^1}{\partial p_g} + \beta \cdot \frac{\partial V^2}{\partial p_g} + \rho \cdot \frac{\partial V^2}{\partial p_g} - \rho \cdot \frac{\partial \hat{V}^2}{\partial p_g} + \mu(N^1 \cdot g^1 + N^2 \cdot g^2) + \mu \cdot p_g [N^1 \cdot \frac{\partial g^1}{\partial p_g} + N^2 \cdot \frac{\partial g^2}{\partial p_g}] \leq 0, \quad \text{and} \quad p_g \left( \frac{\partial \Lambda}{\partial p_g} \right) = 0$$

$$(3-21) \quad \frac{\partial \Lambda}{\partial G} = \frac{\partial V^1}{\partial G} + \beta \cdot \frac{\partial V^2}{\partial G} + \rho \cdot \frac{\partial V^2}{\partial G} - \rho \cdot \frac{\partial \hat{V}^2}{\partial G} + \mu \frac{\partial p_g}{\partial G} (N^1 \cdot g^1 + N^2 \cdot g^2) + \mu \cdot p_g [N^1 \cdot \frac{\partial g^1}{\partial G} + N^2 \cdot \frac{\partial g^2}{\partial G}] - \mu \cdot m = 0$$

---

<sup>17</sup> The Lagrangian problem can also be written with indirect utility function  $V^{*i}$  as:

$$(3-15) \quad \Lambda = V^{*1}(B^1, Y^1, p_v, G) + \beta[V^{*2}(B^2, Y^2, p_v, G) - \bar{V}^{*2}] + \rho[V^{*2}(B^2, Y^2, p_v, G) - V^{*2}(B^1, Y^1, p_v, G)] + \mu[N^2(Y^2 - B^2) + N^1(Y^1 - B^1) + p_v(N^1 \cdot v^1 + N^2 \cdot v^2) - m \cdot G]$$

I will use this Lagrangian problem for the optimal provision rule in Chapter V.

## 2. Desirability and Optimality of Prices for Excludable Public Goods

In order to study the desirability and the optimality of prices for excludable public goods, I will work with equation (3-20), the first order condition with respect to  $p_g$ .

Using Roy's identity and the Slutsky equation, and substituting other first order conditions (3-16) and (3-18), equation (3-20) can be written as

$$(3-20') \quad \frac{\partial \Lambda}{\partial p_g} = \rho \frac{\partial \hat{V}^2}{\partial B^1} [\hat{g}^2 - g^1] + \mu \cdot p_g \sum S^i_{gg} \leq 0, \quad \text{and} \quad p_g \left( \frac{\partial \Lambda}{\partial p_g} \right) = 0,$$

where  $S^i_{gg} < 0$  is the type- $i$  individual's own-price Slutsky term.<sup>18</sup> As Blomquist and Christiansen (2005) pointed out in a similar problem, the first term on the right hand side of the first equation captures the social loss (or benefit) from relaxing the self-selection constraint, while the second term describes a distortionary effect due to the price for a good whose marginal cost is zero.

To discuss the desirability of a positive price, we now have two cases depending on the sign of the first term on the right hand side of the first equation. First, suppose

$\hat{g}^2 \leq g^1$ . Then, any positive price for the public good,  $p_g$ , will make  $\frac{\partial \Lambda}{\partial p_g}$  strictly

negative, which contradicts the second equation,  $p_g \left( \frac{\partial \Lambda}{\partial p_g} \right) = 0$ . Thus, the government

should set a zero price for the public good in this case.

Next, suppose  $\hat{g}^2 > g^1$ . Since the first term on the right hand side of the first equation is strictly positive, we may find a positive  $p_g$  such that  $\frac{\partial \Lambda}{\partial p_g}$  becomes zero.

---

<sup>18</sup> See Appendix 1 for a formal derivation.

Thus, under this condition, a positive price is desirable. An intuitive explanation for this desirability of a positive price is as follows. Consider a small increase in the price for the public good,  $dp_g > 0$ , with a revenue neutral decrease in the income tax. Thus,

$dT(Y^i) = -g^i dp_g$ . Utility for both types of individuals will be unchanged since  $dV^i = -\frac{\partial V^i}{\partial B^i} (g^i dp_g + dT(Y^i)) = 0$  by Roy's Identity. However, the mimicker will be worse off

by this reform, since  $d\hat{V}^2 = -\frac{\partial \hat{V}^2}{\partial B^1} (\hat{g}^2 dp_g + dT(Y^1)) = -\frac{\partial \hat{V}^2}{\partial B^1} (\hat{g}^2 - g^1) dp_g < 0$ . Thus, a positive price when  $\hat{g}^2 > g^1$  makes mimicking less attractive, and allows a relaxation of the self-selection constraint.<sup>19</sup>

The condition that  $\hat{g}^2 > g^1$  means that a type-2 individual, who would mimic a type-1 individual, has a higher demand for public good consumption ( $g$ ) than that of a type-1 individual. Since a mimicker, by definition, takes the bundle  $(B^1, Y^1)$  intended for a type-1 individual, the only difference between two types of individuals is the amount of leisure. In other words, a mimicker would demand more public good consumption only because of more leisure. Therefore, the condition that  $\hat{g}^2 > g^1$  is identical to the condition that marginal valuation of public good consumption ( $g$ ) increases with leisure.

**Proposition 1.** From an initial position with zero  $p_g$ , a positive price for public good is desirable, if and only if marginal valuation of public good consumption ( $g$ ) increases with leisure, that is  $\hat{g}^2 > g^1$ .

**Proof.**

---

<sup>19</sup> This is analogous to the intuition in Edwards et al. (1994).

(→) I have shown that if marginal valuation for public good consumption decreases with or independent of leisure (i.e.  $\hat{g}^2 \leq g^1$ ), then a positive price is *not* desirable. Thus, by contrapositive, if a positive price is desirable, then  $\hat{g}^2 > g^1$ .

(←) When  $\hat{g}^2 > g^1$ , we get  $\left. \frac{\partial \Lambda}{\partial p_g} \right|_{p_g=0} > 0$  from (3-20'), implying a positive price is desirable. **Q.E.D.**

Recall that a price for a public good resembles a commodity tax.<sup>20</sup> Hence, this result with excludable public goods is analogous to the results of Corlett and Hague (1953-1954), Christiansen (1984) and Edwards et al. (1994), that a leisure-complementary good should be taxed along with optimal income taxes.

This proposition also looks very similar to the equation (6) of Blomquist and Christiansen. However, their conclusion that "[t]he policy case for a price may thus appear rather weak" (p. 61) depends on utility initially being unchanged over some interval of  $q$  from  $\{0, \bar{q}\}$ , then falling starting at price  $\bar{q}$ , and then later rising to the socially optimal  $q$ . That is, the uniqueness of their modeling, which earlier in Chapter III turned out to be too restrictive, results in the implication of a weaker policy case. With a more reasonable characterization of excludable public goods developed here, I have shown that utility unambiguously increases as the price is increased starting at zero. Thus, I come to find a stronger policy case to be made for pricing excludable public goods.

Under the same condition of Proposition 1, the optimal pricing rule for an excludable public good can be easily derived.

---

<sup>20</sup> See pp. 8-9 for more explanation.

**Proposition 2.** (Optimal Pricing Rule)

$$\text{When } \hat{g}^2 > g^1, p_g^* = \frac{\partial \hat{V}^2}{\partial B^1} \frac{\rho[g^1 - \hat{g}^2]}{\mu \sum S^i_{gg}}$$

**Proof.** It can be derived just by rearranging equation (3-20').

**Q.E.D.**

Recall that the condition that  $\hat{g}^2 > g^1$  is identical to the condition that marginal valuation of public good consumption ( $g$ ) increases with leisure. For further analysis in the next chapter, I make sure that the condition is also the same as the condition that marginal valuation of public good itself ( $G$ ) increases with leisure. Hence, I here briefly discuss the relationships between two marginal valuations.<sup>21</sup>

**Proposition 3.**  $\hat{g}^2 > g^1$  if and only if  $M\hat{R}S^2_{GB} > MRS^1_{GB}$

**Proof.**

First note that the condition that  $\hat{g}^2 > g^1$  is identical to the condition that a mimicker values the public good consumption more than a type-1 individual. Since a mimicker and a type-1 individual have the same disposable income, we get  $\hat{c}^2 < c^1$ . Therefore,  $\hat{g}^2 > g^1$  if and only if  $M\hat{R}S^2_{gc} > MRS^1_{gc}$  when evaluated at the same ( $g, c, B$ ).

( $\rightarrow$ ) From the definition of  $MRS_{Gc}$  which implicitly assumes that other things are equal

including  $v$ , we may fix  $v$  and normalize it at unity. Thus,  $\frac{\partial \hat{U}^2}{\partial g} / \frac{\partial \hat{U}^2}{\partial c} > \frac{\partial U^1}{\partial g} / \frac{\partial U^1}{\partial c}$

implies  $\frac{\partial \hat{U}^2}{\partial G} / \frac{\partial \hat{U}^2}{\partial c} > \frac{\partial U^1}{\partial G} / \frac{\partial U^1}{\partial c}$ . By the envelop theorem, we get  $\frac{\partial V}{\partial G} = \frac{\partial U}{\partial G}$ . Also, it is

easy to show that  $\frac{\partial V}{\partial B} = \frac{\partial U}{\partial c}$  because the private good is the numeraire. Hence,

$\frac{\partial \hat{U}^2}{\partial G} / \frac{\partial \hat{U}^2}{\partial c} > \frac{\partial U^1}{\partial G} / \frac{\partial U^1}{\partial c}$  is identical to  $\frac{\partial \hat{V}^2}{\partial G} / \frac{\partial \hat{V}^2}{\partial B} > \frac{\partial V^1}{\partial G} / \frac{\partial V^1}{\partial B}$ , or  $M\hat{R}S^2_{GB} > MRS^1_{GB}$ .

( $\leftarrow$ ) By the same logic as above,  $\hat{g}^2 < g^1$  implies  $M\hat{R}S^2_{GB} < MRS^1_{GB}$ ; and  $\hat{g}^2 = g^1$  implies  $M\hat{R}S^2_{GB} = MRS^1_{GB}$ . Thus, by contrapositive, this means that  $M\hat{R}S^2_{GB} > MRS^1_{GB}$  implies  $\hat{g}^2 > g^1$ . **Q.E.D.**

---

<sup>21</sup> Marginal valuation can be described by the marginal rate of substitution between two variables :  $MRS^i_{xy} = \frac{\partial U^i}{\partial x} / \frac{\partial U^i}{\partial y}$ .

Before proceeding, I verify how this result works with weakly separable utility between consumption and leisure, which has been studied in the literature on a nonlinear income tax model including Stiglitz (1982), Christiansen (1984), Boadway and Keen (1993) and Edwards et al.(1994).

**Corollary 1.** If utility is weakly separable between consumption and leisure (i.e.  $U^i(c^i, g^i, H^i) = U^i(F(c^i, g^i), H^i)$ ), then  $p_g^* = 0$ .

**Proof.** If utility is weakly separable between consumption and leisure, then the marginal rate of substitution between  $g^i$  and  $c^i$  is independent of  $H^i$ .<sup>22</sup> Therefore,  $M\hat{R}S_{gc}^2 = MRS_{gc}^1$  because the only difference between a mimicker and a type-1 individual is  $H^i$  (or leisure). Since a mimicker and a type-1 individual have the same income and the same preferences over public and private goods, the allocation of consumption should also be the same:  $\hat{g}^2 = g^1$ . Thus, from Proposition 1, the optimal price is zero. **Q.E.D.**

So far in this section, I have studied that a positive price for an excludable public good should be implemented when the marginal valuation of the public good increases with leisure in a second-best economy where the government considers redistribution within the society. The next section will be devoted to derivation of the optimal provision rule by modifying Samuelson Condition.

### 3. Optimal Provision Rule for Excludable Public Goods

Having derived the optimal pricing rule for excludable public goods in the last section, I turn to deriving an optimal provision rule for excludable public goods, a version of a modified Samuelson Condition.

---

<sup>22</sup> See Goldman and Uzawa (1964). They showed that if a utility function is weakly separated by two or more sub-utility functions in it, the marginal rates of substitution between any two variables in a sub-utility function is independent of a variable in another sub-utility function.

**Proposition 4.** (Optimal Provision Rule)

$$\sum MRS^i_{GB} = m + \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^2}{\partial B^1} \cdot [MR\hat{S}^2_{GB} - MRS^1_{GB}] - p_v \sum \frac{\partial v^{i^c}}{\partial G}$$

**Proof.** See Appendix 2.

It is not surprising that this proposition is exactly analogous to Proposition 3 in Edwards et al. (1994), because a positive price for public good is a kind of commodity tax. Note that the Samuelson Condition would imply  $\sum MRS^i_{GB} = m$ , assuming the first-best economy. Thus, in order to compare this equation with the Samuelson Condition, we need to discuss two additional terms of the equation given in Proposition 4.

The first term,  $\frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^2}{\partial B} \cdot [MR\hat{S}^2_{GB} - MRS^1_{GB}]$ , captures the effect of mimicking

incentives on the total social cost. This term will be referred as to the “mimicking term.” I assume momentarily that the level of access to the public good is unchanged with  $G$  :

$\frac{\partial v^{i^c}}{\partial G} = 0$ . Suppose a mimicker values the public good more than a type-1 individual does,

that is,  $MR\hat{S}^2_{GB} > MRS^1_{GB}$ . Then,  $\sum MRS^i_{GB} > m$  from Proposition 4. If  $\sum MRS^i_{GB}$  were monotonically decreasing in  $G$ , this second-best optimum that  $\sum MRS^i_{GB} > m$  would imply an underprovision of the public good relative to the first-best optimum.<sup>23</sup> Now, consider an initial situation where the Samuelson Condition holds. The government decreases  $G$  marginally with revenue unchanged by lowering income taxes of each type of individual by the amount of their  $MRS^i_{GB}$ . Then, there is no change in utility of either

---

<sup>23</sup> The notion of “under” provision is only for intuitive explanations. I will study in detail “over” and “under” provision in Chapter V.

type-1 or type-2 individuals; the government revenue is also unchanged. However, this reform makes a mimicker worse off because he loses  $M\hat{R}S^2_{GB}$  in the form of the public good, and gains only  $MRS^1_{GB}$  in the form of income tax. Thus, the self-selection constraints can be relaxed, which makes this economy Pareto-optimal. This effect is first reported by Boadway and Keen with a non-excludable public good financed by income tax, and then by Edwards et al. with a non-excludable public good financed by both direct and indirect tax.

The second term  $p_v \sum \frac{\partial v^{i^c}}{\partial G}$  describes the revenue effect. Suppose  $\sum \frac{\partial v^{i^c}}{\partial G}$  is

positive. With  $G$  increased, the government can gather additional revenue based on the increased amount of access to the public good. Note that this measure is associated with Hicksian (compensated) demand. This is because when we consider the same reform of increasing  $G$  financed by the sum of  $MRS^i_{GB}$  as discussed above, utility for each type of individual is unchanged. Hence only compensated demand should be taken account.

In this model, the revenue term can dominate the mimicking term depending on the specific utility function, thus this optimal rule is another result proving that Pigou's conjecture that distortionary taxation will increase the total social cost is not true. Furthermore, the weak separability that has been discussed in the last section implies that the Samuelson Condition still holds, even in a second-best economy.

**Corollary 2.** If utility is weakly separable between consumption and leisure (i.e.  $U^i(c^i, g^i, H^i) = U^i(F(c^i, g^i), H^i)$ ), the Samuelson Condition holds:  $\sum MRS^i_{GB} = m$ .

**Proof.** From Corollary 1, with this type of utility, the optimal price for the public good is zero. Thus, the revenue effect vanishes. Also, a weakly separable utility function is  $U^i(F(c^i, g^i), H^i) = U^i(Q(c^i, v^i, G), H^i)$ , so  $M\hat{R}S^2_{Gc} = MRS^1_{Gc}$  since the only difference between a mimicker and a type-1 individual is the amount of leisure ( $H^i$ ). This also implies  $M\hat{R}S^2_{GB} = MRS^1_{GB}$ . Thus, the mimicking term also vanishes. **Q.E.D.**

Now, I revisit the discussion of the flexibility of this public good model that we once discussed in Chapter III where I showed that the model can be used for non-excludable public goods simply by setting  $v$  as a constant (i.e. “national defense” for case 1) or by putting  $p_v$  at zero (i.e. “fresh air” for case 2). We can easily confirm the flexibility again with the equation of Proposition 4:

$$\sum MRS^i_{GB} = m + \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^2}{\partial B^1} \cdot [M\hat{R}S^2_{GB} - MRS^1_{GB}] - p_v \sum \frac{\partial v^{i^c}}{\partial G}.$$

The revenue term,  $p_v \sum \frac{\partial v^{i^c}}{\partial G}$ , disappears either when  $p_v$  is equal to zero, or when  $v$  is fixed, so that the equation becomes exactly the same as equation (9) in Boadway and Keen, the optimal provision rule for non-excludable public goods. Hence, this is another way to see the applicability of our model to non-excludable cases.

I have derived the optimal pricing rule and provision rule for excludable public goods in a second-best framework in this chapter. However, I have not yet addressed the quantity question about the optimal size of excludable public goods. In the next chapter, I will study these questions and derive sufficient conditions that can guarantee overprovision or underprovision, relative to the first-best optimal size.

## V. Optimal Provision Level Comparison

Having identified the optimal pricing and provision rules in the last chapter, I now turn to the comparison of the optimal provision level for excludable public goods between the first-best economy and the second-best economy discussed in this thesis. More specifically, I will derive sufficient conditions where an excludable public good in the second-best economy is underprovided (or overprovided) relative to the first-best economy. The planner's optimization problem in this second-best economy is:<sup>24</sup>

$$(3-10') \quad \text{Max}_{B^1, Y^1, B^2, Y^2, p_v, G} V^{*2}(B^2, Y^2, p_v, G) \quad \text{subject to}$$

$$(3-11') \quad V^{*1}(B^1, Y^1, p_v, G) \geq \bar{V}^{*1}$$

$$(3-12') \quad V^{*2}(B^2, Y^2, p_v, G) \geq \hat{V}^{*2}(B^1, Y^1, p_v, G)$$

$$(3-13') \quad N^2 \cdot (Y^2 - B^2) + N^1 \cdot (Y^1 - B^1) + p_v \cdot (N^1 \cdot v^1 + N^2 \cdot v^2) - m \cdot G \geq 0.$$

The reference to the optimal provision level  $G$  from this maximization will be  $G^S$  from this point forward.  $(B^{1S}, Y^{1S}, B^{2S}, Y^{2S}, G^S)$  is the corresponding allocation.

In the first-best economy, the government can distinguish the type of each individual and implement lump-sum tax on each type of individual with distortionary prices for the public good being zero. Thus, the optimization problem in the first-best economy is:

$$(F-1) \quad \text{Max}_{B^1, Y^1, B^2, Y^2, G} V^{*2}(B^2, Y^2, 0, G) \quad \text{subject to}$$

$$(F-2) \quad V^{*1}(B^1, Y^1, 0, G) \geq \bar{V}^{*1}$$

$$(F-3) \quad N^2 \cdot (Y^2 - B^2) + N^1 \cdot (Y^1 - B^1) - m \cdot G \geq 0.$$

---

<sup>24</sup> The problem setting here has two differences from the original problem setting in the previous chapters. First, here I use  $V^{*i}$  defined with  $p_v$ , unlike  $V^i$  with  $p_g$ . However, it is easily shown that whether  $V^i$  or  $V^{*i}$  is used does not make any difference. Second, here I maximize the utility of a type-2 individual, not a type-1 individual. Since the selection of the type of utility to be maximized is arbitrary, one can use either one. However, I use this setting here to make sure all constraints are binding, as Stiglitz reports that constraints (3-11') and (F-2) are always binding.

The optimal allocation derived from solving this first-best problem is  $(B^{1F}, Y^{1F}, B^{2F}, Y^{2F}, G^F)$ .

In order to compare  $G^S$  and  $G^F$ , I follow the method of Gaube, who decomposes changes from  $G^S$  to  $G^F$  into two pieces: ‘substitution effect’ and ‘income effect.’ I therefore introduce an intermediate economy, which is described as:

$$(I-1) \quad \text{Min}_{B^1, Y^1, B^2, Y^2, G} \quad m \cdot G - N^2 \cdot (Y^2 - B^2) - N^1 \cdot (Y^1 - B^1) \quad \text{subject to}$$

$$(I-2) \quad V^{*1}(B^1, Y^1, 0, G) \geq \bar{V}^{*1}$$

$$(I-3) \quad V^{*2}(B^2, Y^2, 0, G) \geq \bar{V}^{*2}.$$

For this intermediate economy, the optimal allocation is  $(B^{1Z}, Y^{1Z}, B^{2Z}, Y^{2Z}, G^Z)$ . It would be useful to acknowledge important properties of this intermediate economy. Since the two constraints (I-2) and (I-3) must be binding, the utility levels remain the same as those of the second-best economy. However, the optimal provision rule in this intermediate economy is exactly identical to that of the first-best economy, because the planner does not have to be worried about mimicking behavior. These two properties explain how one can decompose changes from  $G^S$  to  $G^F$  into two pieces:  $(G^S$  to  $G^Z)$  and  $(G^Z$  to  $G^F)$ . First, the changes from  $G^S$  to  $G^Z$  are due to slackening the self-selection constraint, that is, changing implicit relative prices for public goods, with the utility level unchanged. Thus, this is analogous to a substitution effect. Second, the changes from  $G^Z$  to  $G^F$  are due to increasing utility with all other conditions unchanged, which is a similar concept to an income effect. Using these two effects, I will derive conditions where  $G^F$  is greater than  $G^S$  (underprovision), and vice versa (overprovision).

Since several propositions derived in Stiglitz and Gaube play important roles here, they should be summarized before proceeding.

**Fact 1** (Stiglitz (1982)) in a second-best economy,<sup>25</sup>

- ① The marginal tax rate of the high-ability type is zero
- ② The marginal tax rate of the low-ability type is positive
- ③ The constraint (3-11') is binding.

**Fact 2** (Gaube (2005)) in a second-best economy,<sup>26</sup>

$$\sum MRS_{GY}^i{}^S > \sum MRS_{GB}^i{}^S$$

The first two results of Fact 1 affects an individual's decision on labor supply in a second-best economy. A type-*i* individual will choose his labor supply level to make the marginal rate of substitution between labor and consumption equal to his effective after-tax wage rate. However, since the marginal tax rate in the first-best economy is zero for each type of individual, these results make a difference in a low-type individual's labor supply decision between a second-best economy and the first-best economy. The third result plays a role of ensuring that the first-best economy is Pareto-dominating a second-best economy, as I will discuss later.

To illustrate Fact 2, I need to briefly discuss the matter of the numeraire. As Atkinson and Stern (1974) and Boadway and Keen (1993) pointed out, choices of the numeraire may produce different results of optimal provision rules. In this nonlinear income tax model, one can alternatively choose the pre-tax income  $Y^i$  (or equivalently

labor  $L^i$ ) as the numeraire. Then, besides  $MRS_{GB} = \frac{\partial V^i}{\partial G} / \frac{\partial V^i}{\partial B^i}$ , I define another

---

<sup>25</sup> Although Stiglitz considers neither commodity tax nor excludable public goods, it is easy to see that the intuition and analysis behind these propositions can be exactly applicable, with little modification, to the analysis with a positive price for public goods.

<sup>26</sup> In the chapter, the superscription "s" of  $MRS$  will be used to indicate a second-best economy.

marginal rate of substitution, between  $G$  and  $Y$ , accordingly:  $MRS_{GY} = \frac{\partial V^i}{\partial G} / \frac{\partial V^i}{\partial Y^i}$ .

Since Gaube has shown that  $\sum MRS_{GY}^i > \sum MRS_{GB}^i$ , I may have three possible cases:

first,  $\sum MRS_{GY}^i > \sum MRS_{GB}^i \geq m$ ; second,  $\sum MRS_{GY}^i \geq m \geq \sum MRS_{GB}^i$ ; third,  $m \geq$

$\sum MRS_{GY}^i > \sum MRS_{GB}^i$ . One can see that the matter of the numeraire is one of the

reasons why optimal provision rules cannot directly imply the relative optimal level. In

the second case, even though the optimal provision rule with the numeraire of  $B$  may

imply overprovision (i.e.  $m \geq \sum MRS_{GB}^i$ ), it may well be the other way around if we

treat  $Y$  as the numeraire (i.e.  $\sum MRS_{GY}^i \geq m$ ). As the main goal here is to derive

sufficient conditions for underprovision and overprovision, I will focus on the first and

third case, respectively.

Finally, considerations will only be given to the situation where a positive price is optimal (i.e.  $\hat{MRS}_{GB}^2 \geq MRS_{GB}^1$ ). Otherwise, the analysis of this public good is exactly the same as that of a non-excludable public good, which has already been studied in Gaube (2005).<sup>27</sup> Not surprisingly, this will lead to a major difference in relative possibilities for underprovision or overprovision between excludable public goods and non-excludable public goods, as I will discuss in the final section.

---

<sup>27</sup> I have already shown that the optimal price for public good is positive if and only if  $\hat{MRS}_{GB}^2 \geq MRS_{GB}^1$ . See the Proposition 3.

## 1. A Sufficient Condition for Underprovision

Intuitively, public goods are under-provided when the sum of the marginal rate of substitution is greater than the marginal rate of transformation with *each* of the numeraire, because marginal benefit generally decreases with the amount of  $G$ . Thus, one might argue that when  $\sum MRS^i_{GY^S} > \sum MRS^i_{GB^S} \geq m$ , public goods are underprovided relative to the first-best level. However, more care needs to be taken with conditions for underprovision, since  $B$  (or private good consumption) and  $Y$  (or labor) must be related to  $G$  so that the marginal rates of substitution might not always be decreasing with  $G$ .

In order to derive sufficient conditions of underprovision (i.e.  $G^S < G^F$ ), I decompose the transition ( $G^S$  to  $G^F$ ) into two steps: the first step ( $G^S$  to  $G^Z$ ) and the second step ( $G^Z$  to  $G^F$ ). I then focus on the conditions that guarantee that  $G$  should increase for both steps (i.e.  $G^S < G^Z$  and  $G^Z < G^F$ ).<sup>28</sup>

**Proposition 5.**  $\sum MRS^i_{GB^S} \geq m$  implies  $G^S < G^F$  as long as following three conditions are satisfied:

- ① the public good and leisure are Hicksian substitutes
- ② the public good and access to the public good are not Hicksian substitutes
- ③ the public good is normal

**Proof.** See Appendix 3.

The first two conditions are needed for the first step, which is  $G^S < G^Z$ . The transition from the second-best economy to the intermediate economy makes a positive marginal tax rate for a low-type individual decrease to zero. This results in an increase in

---

<sup>28</sup> Derivation of necessary conditions involves the comparison of the amount of substitution effect with that of income effect, which is beyond the scope of this thesis.

the net wage rate. In other words, the relative leisure price for a low-type individual increases due to this transition. Thus, when the public good and leisure are substitutes for each other, the demand for the public good must increase due to this transition.<sup>29</sup>

Next, the explanation of the second condition is as follows. With other things being equal, if the access to the public good is a Hicksian substitute for the public good itself, the public good demand  $G^Z$  with a zero “access” price will be lower than  $G^S$  with a positive “access” price, which cannot guarantee underprovision,  $G^S < G^Z$ . Thus, the second condition above should hold for this sufficient condition.

Finally, the third condition of the public good being normal involves the second step,  $G^Z \leq G^F$ . Note that the utility level in the first-best economy is not less than that of the intermediate economy, i.e.  $U^{iZ} \leq U^{iF}$ ,<sup>30</sup> and that other properties such as relative prices are the same across these two economies. In other words, all of the prices are identical, but only the expenditure (or income) level to attain the utility level is higher in the first-best economy than in the intermediate economy. This makes the first-best Hicksian demand for public good larger than that of intermediate economy as long as the public good is normal.

---

<sup>29</sup> Explanations of the first condition are identical to those in Gaube as quoted:  
*“...the transition from second best to (9) can be interpreted as the consequence of a decreasing price of the public good, which (under the assumption of constant utility levels) increases the households’ demand. However, this transition also affects the relative price between leisure and private consumption. This is due to the fact that the marginal income tax of the low-ability type is positive in second best while it is zero in the optimum (9). The increasing net wage of the low-ability households raises their (compensated) demand for the public good only if leisure and the public good are Hicksian substitutes. Therefore, if these two commodities are Hicksian complements, the total effect – may become negative.”*

<sup>30</sup> See Appendix 5.

Hence, when a positive price is optimal for an excludable public good, the condition that  $\sum MRS^i_{GY^S} > \sum MRS^i_{GB^S} \geq m$  would imply underprovision of the public good relative to the first-best level, although it still needs two additional assumptions.

## 2. A Sufficient Condition for Overprovision

When a public good is excludable, it is possible that the sum of the marginal rate of substitution is less than the marginal rate of transformation, even when a positive optimal price should be implemented (i.e.  $\hat{MRS}^2_{GB} \geq MRS^1_{GB}$ ). That is, we may well get the condition  $m \geq \sum MRS^i_{GY^S} > \sum MRS^i_{GB^S}$ .<sup>31</sup> In this case, it can be shown that the optimal level of public good provision in the second-best economy would be bigger than that of the first-best economy if three more conditions are satisfied.

**Proposition 6.**  $\sum MRS^i_{GY^S} \leq m$  implies  $G^S > G^F$  as long as following three conditions are satisfied:

- ① the public good and the private good are Hicksian substitutes
- ② the public good and access to the public good are not Hicksian complements
- ③ the public good is not normal

**Proof.** See Appendix 4.

The interpretation of these three conditions is similar to those in the previous section. As in the previous section, the first two conditions are associated with the

---

<sup>31</sup> Recall that the modified Samuelson condition is  $\sum MRS^i_{GB^S} = m + \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^2}{\partial B} \cdot [M\hat{R}S^2_{GB} - MRS^1_{GB}] - \rho_v \sum \frac{\partial v^i}{\partial G}$ .

Now, the absolute value of the third term on the right hand side may be greater than that of the second term. This issue will be discussed in detail in the next section.

transition from our second-best economy to the intermediate economy, implying  $G^S > G^Z$ , while the third condition is for the condition that  $G^Z \geq G^F$ .

I start with the first condition. The transition makes the net wage rate, or the relative leisure price, higher as explained in the last section. One might think of this effect as a decrease in the relative private good price, if labor is considered to be the numeraire.<sup>32</sup> Therefore, to guarantee the public good demand a decrease due to this transition, private consumption as a Hicksian substitute for the public good should be one assumption for this sufficient condition. Secondly, the transition from the second-best economy to the intermediate economy also involves a decrease in the price for the access to the public good. Thus, if the public good and the access to it are Hicksian complements, demand for the public good would increase as  $p_v$  decreases, so we cannot guarantee that  $G^S > G^Z$ . Finally, if the public good is normal, a higher utility level (or a corresponding higher income level) due to the transition from the intermediate economy to the first-best economy would imply a higher demand for the public good, so again we cannot guarantee that  $G^Z \geq G^F$ . Therefore, each of the three conditions given in Proposition 6 should be part of the sufficient condition for overprovision.

So far, I have derived sufficient conditions for underprovision and overprovision of excludable public goods in the second-best economy characterized throughout this thesis. Since I follow the approach of Gaube (2005), these sufficient conditions are analogous to those of Gaube. Furthermore, when we set  $v$  constant to make the analyses on non-excludable public goods, each of the second conditions in Proposition 5 and 6 are

---

<sup>32</sup> In the previous section, I treated this effect as an increase of leisure price, which is the net wage rate with private consumption (or equivalently after-tax income) being the numeraire.

automatically satisfied so that the propositions are identical to those of Gaube. I now turn to discussions of differences between excludable public goods and non-excludable public goods in terms of the optimal provision level of public goods.

### 3. Effect of Excludability on Public Good Provision Level

Recall that, in this chapter, I have focused on the case that  $\hat{MRS}^2_{GB} \geq MRS^1_{GB}$  in order to guarantee an optimal positive price for an excludable public good. For non-excludable public goods as in Gaube, the condition  $\hat{MRS}^2_{GB} \geq MRS^1_{GB}$  is identical to the condition  $\sum MRS^i_{GB}^S \geq m$ , which is a sufficient condition for *underprovision* if supplemented by two proper conditions. However, when the public goods are excludable, the condition  $\hat{MRS}^2_{GB} \geq MRS^1_{GB}$  does not necessarily guarantee  $\sum MRS^i_{GB}^S \geq m$ . Thus, from the modified Samuelson Condition with  $B$  being the numeraire,

$$\sum MRS^i_{GB}^S = m + \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^2}{\partial B} \cdot [\hat{MRS}^2_{GB} - MRS^1_{GB}] - \tilde{p}_v \sum \frac{\partial v^{i^c}}{\partial G},$$

it is necessary to discuss whether or not the revenue term dominates the mimicking term.

When the mimicking term  $\frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^2}{\partial B} \cdot [\hat{MRS}^2_{GB} - MRS^1_{GB}]$  dominates, the condition

$\hat{MRS}^2_{GB} \geq MRS^1_{GB}$  guarantees that  $\sum MRS^i_{GB}^S$  should be greater than the cost  $m$  which implies underprovision, assuming the other three conditions of Proposition 5 are satisfied.

On the other hand, if the revenue term dominates, then  $\sum MRS^i_{GB}^S$  is less than  $m$ , so that we may well have overprovision of a public good, assuming the three conditions given in Proposition 6 are satisfied. This is summarized in Table 2.

Table 2. Conditions for Underprovision and Overprovision

Conditions	$M\hat{R}S^2_{GB} \geq MRS^1_{GB}$	$M\hat{R}S^2_{GY} < MRS^1_{GY}$
<i>optimal price for excludable public goods</i>	<i>positive</i>	<i>zero</i> <sup>33</sup>
Non-Excludable	Underprovision	Overprovision
Excludable	Case 1. Underprovision (when the “mimicking” term dominates)  Case 2. Overprovision (when the “revenue” term dominates)	

(Note: all are cases with proper assumptions given in Gaube or in Proposition 5 and 6)

Note that the questions of which term dominates (i.e. the mimicking term or the revenue term) and which conditions are more reasonable (i.e. conditions of Proposition 5 or 6) depend on characteristics of particular public goods and of individual utility functions. However, one can see that there may be overprovision even when a positive price for an excludable public good is optimal (i.e.  $M\hat{R}S^2_{GB} \geq MRS^1_{GB}$ ), and that underprovision cases of excludable public goods that can be found in a real world are smaller than those of non-excludable public goods.<sup>34</sup> Thus, the argument of Blomquist and Christiansen (2005) that the optimal provision level will be smaller in a second-best economy when a mimicker values public good consumption more than a low-type individual, again depends in a crucial way on their characterization of excludable public goods, and as such is misleading.

<sup>33</sup> It can be shown that  $M\hat{R}S^2_{GY} < MRS^1_{GY}$  implies  $M\hat{R}S^2_{GB} < MRS^1_{GB}$ , which is the condition for a zero price for the excludable public good. See Appendix 6 for the proof.

<sup>34</sup> Note that, if  $M\hat{R}S^2_{GB} < MRS^1_{GB}$ , the planner will impose zero prices, which makes the situation the same as the non-excludable public good case.

## **VI. Conclusion**

I have studied the optimal rules and the provision level for excludable public goods using a nonlinear income tax model with a self-selection approach. When the government is willing to make a redistribution policy, but cannot observe individuals' abilities or labor supply, I have shown that a positive price for a public good is desirable provided that marginal valuation of the public good increases with leisure. This result is one of the points that links this research to the literature on indirect taxes supplementing optimal income taxation, such as Corlett and Hague and Edwards et al. I have also derived a modified Samuelson Condition for excludable public goods due to the self-selection and the excludability of public goods, characterized by the mimicking term and the revenue term. While the mimicking term comes from the self-selection constraint set by the government in order to prevent mimicking behaviors, the revenue term describes additional revenue effects of the positive price for excludable public goods. Note that this revenue term is also found in the literature on optimal provision rules for public goods with indirect taxation. Thus, this is another point where one might find similarities between this research and other literature on optimal income taxation supplemented by indirect taxes. Therefore, I believe one of the main contributions of this thesis is that the model of excludable public goods I have developed provides firm links between prices for excludable public goods and excise taxes on private goods. Since there is little formal difference between them, one may apply many of the results from indirect taxation research to the excludable public good cases.

Another contribution of this thesis is that the model of public goods that I have developed here provides a general model for pure public goods as well as excludable

public goods, and that it can be applicable to a large number of public good examples. Hence, one can use this model for future research on a public good to analyze the impact of excludability of the public good by, for example, first imposing a restriction of the price being equal to zero and then relaxing it.

Further research can follow this thesis in various ways. First, for the characterization of public goods, I have precluded the possibility of congestion, but it may be more reasonable to allow a congested situation upon satisfaction of certain conditions (i.e. the number of consumers hits a certain level), for some public goods. Thus, one can study a two-stage problem with an uncongested stage first and then with a congested stage.

Second, more studies should be done with different pricing schemes. The implicit assumption of the unit-price scheme in this thesis might have limited the applicability of the public good model. An all-or-nothing pricing scheme or a mixed scheme, such as a two-part tariff, may be more reasonable for certain public goods, and can consequently be used for pricing schemes in the future excludable public good research.

Finally, in addition to the theoretical aspects, I have to further explore empirical studies regarding the sufficient conditions for the optimal size of excludable public goods discussed in Chapter V. That is, it is important to empirically determine the characteristics of a given public good (i.e. whether or not the public good is a Hicksian substitute of leisure), so that we can get clearer results about whether a public good would be overprovided or underprovided in the second-best optimum.

## Appendices

### Appendix 1: derivation of equation (3-20') out of equations (3-16) to (3-20)

Recall that the equations are:

$$(3-16) \quad \frac{\partial \Lambda}{\partial B^1} = \frac{\partial V^1}{\partial B^1} - \rho \cdot \frac{\partial \hat{V}^2}{\partial B^1} - \mu \cdot N^1 + \mu \cdot p_g \cdot N^1 \cdot \frac{\partial g^1}{\partial B^1} = 0$$

$$(3-17) \quad \frac{\partial \Lambda}{\partial Y^1} = \frac{\partial V^1}{\partial Y^1} - \rho \cdot \frac{\partial \hat{V}^2}{\partial Y^1} + \mu \cdot N^1 + \mu \cdot p_g \cdot N^1 \cdot \frac{\partial g^1}{\partial Y^1} = 0$$

$$(3-18) \quad \frac{\partial \Lambda}{\partial B^2} = \beta \cdot \frac{\partial V^2}{\partial B^2} + \rho \cdot \frac{\partial V^2}{\partial B^2} - \mu \cdot N^2 + \mu \cdot p_g \cdot N^2 \cdot \frac{\partial g^2}{\partial B^2} = 0$$

$$(3-19) \quad \frac{\partial \Lambda}{\partial Y^2} = \beta \cdot \frac{\partial V^2}{\partial Y^2} + \rho \cdot \frac{\partial V^2}{\partial Y^2} + \mu \cdot N^2 + \mu \cdot p_g \cdot N^2 \cdot \frac{\partial g^2}{\partial Y^2} = 0$$

$$(3-20) \quad \frac{\partial \Lambda}{\partial p_g} = \frac{\partial V^1}{\partial p_g} + \beta \cdot \frac{\partial V^2}{\partial p_g} + \rho \cdot \frac{\partial V^2}{\partial p_g} - \rho \cdot \frac{\partial \hat{V}^2}{\partial p_g} + \mu(N^1 \cdot g^1 + N^2 \cdot g^2) \\ + \mu \cdot p_g [N_1 \cdot \frac{\partial g^1}{\partial p_g} + N_2 \cdot \frac{\partial g^2}{\partial p_g}] \leq 0, \quad p_g \left( \frac{\partial \Lambda}{\partial p_g} \right) = 0.$$

First, from  $V^i(B^i, Y^i, p_g, G) = \max_{c^i, g^i} U\left(c^i, g^i, \frac{Y^i}{w^i}\right) + \lambda(B^i - c^i - p_g g^i)$ , the

envelope theorem implies

$$(A-1) \quad \frac{\partial V^i}{\partial p_g} = -\lambda g^i = -\frac{\partial V^i}{\partial B^i} g^i.$$

Using the Slutsky equation for  $g$ , that is,

$$(A-2) \quad S_{gg}^i = \frac{\partial g^i}{\partial p_g} + g^i \frac{\partial g^i}{\partial B^i},$$

equation (3-20) becomes:

$$\begin{aligned}
\text{(A-3)} \quad \frac{\partial \Lambda}{\partial p_g} &= \frac{\partial V^1}{\partial p_g} + \beta \cdot \frac{\partial V^2}{\partial p_g} + \rho \cdot \frac{\partial V^2}{\partial p_g} - \rho \cdot \frac{\partial \hat{V}^2}{\partial p_g} + \mu(N^1 \cdot g^1 + N^2 \cdot g^2) \\
&\quad + \mu \cdot p_g [N_1 \cdot \frac{\partial g^1}{\partial p_g} + N_2 \cdot \frac{\partial g^2}{\partial p_g}] \\
&= -\frac{\partial V^1}{\partial B^1} g^1 - \beta \cdot \frac{\partial V^2}{\partial B^1} g^2 - \rho \cdot \frac{\partial V^2}{\partial B^2} g^2 + \rho \cdot \frac{\partial \hat{V}^2}{\partial B^1} \hat{g}^2 + \mu(N^1 \cdot g^1 + N^2 \cdot g^2) + \\
&\quad \mu \cdot p_g [N^1 \cdot (S^1_{gg} - g^1 \frac{\partial g^1}{\partial B^1}) + N^2 \cdot (S^2_{gg} - g^2 \frac{\partial g^2}{\partial B^2})].
\end{aligned}$$

Adding and subtracting  $\rho \cdot g^1 \frac{\partial \hat{V}^2}{\partial B^1}$ , (A-3) becomes:

$$\begin{aligned}
\text{(A-3')} \quad \frac{\partial \Lambda}{\partial p_g} &= -\frac{\partial V^1}{\partial B^1} g^1 - \beta \cdot \frac{\partial V^2}{\partial B^1} g^2 - \rho \cdot \frac{\partial V^2}{\partial B^2} g^2 + \rho \cdot \frac{\partial \hat{V}^2}{\partial B^1} \hat{g}^2 + \rho \cdot g^1 \frac{\partial \hat{V}^2}{\partial B^1} - \rho \cdot g^1 \frac{\partial \hat{V}^2}{\partial B^1} \\
&\quad + \mu(N^1 \cdot g^1 + N^2 \cdot g^2) + \mu \cdot p_g [N^1 \cdot (S^1_{gg} - g^1 \frac{\partial g^1}{\partial B^1}) + N^2 \cdot (S^2_{gg} - g^2 \frac{\partial g^2}{\partial B^2})] \\
&= \rho \cdot \frac{\partial \hat{V}^2}{\partial B^1} \hat{g}^2 - \rho \cdot g^1 \frac{\partial \hat{V}^2}{\partial B^1} + \mu \cdot p_g [N^1 \cdot (S^1_{gg}) + N^2 \cdot (S^2_{gg})].
\end{aligned}$$

The last equality follows directly from applying FOC conditions (3-16) and (3-18) to the

equation. Therefore,  $\frac{\partial \Lambda}{\partial p_g} = \rho \frac{\partial \hat{V}^2}{\partial B^1} [\hat{g}^2 - g^1] + \mu \cdot p_g \sum S^i_{gg}$  **Q.E.D.**

## Appendix 2: proof of Proposition 4

First, from the alternative indirect utility of  $V^{*i}(B^i, Y^i, p_v, G)$  of equation (3-7)

on p. 23, *MRS* is defined as

$$\text{(A-4)} \quad MRS^i_{GB} = \frac{\partial V^{*i}}{\partial G} / \frac{\partial V^{*i}}{\partial B^i} \quad \text{with } B \text{ being the numeraire.}$$

Second, let us verify the Slutsky equation for  $v$  (the access to the public good) with change in  $G$ . Consider the identity,

$$(A-5) \quad v^{i^c}(\bar{u}_i, p_v, G) = v^i(e^i(p_v, G, w^i, \bar{u}^i), p_v, G).$$

Taking derivatives with respect to  $G$ , we get

$$(A-6) \quad \frac{\partial v^{i^c}}{\partial G} = \frac{\partial v^i}{\partial G} + \frac{\partial v^i}{\partial B^i} \cdot \frac{\partial e^i}{\partial G}.$$

Since  $\frac{\partial e^i}{\partial G}$  describes changes in how much income is needed to keep the utility remaining

unchanged when  $G$  gets higher, that is,

$$(A-7) \quad \frac{\partial e^i}{\partial G} = -\frac{\partial V^{*i}}{\partial G} / \frac{\partial V^{*i}}{\partial B^i} = -MRS^i_{GB},$$

the equation (A-6) becomes:

$$(A-6') \quad \frac{\partial v^{i^c}}{\partial G} = \frac{\partial v^i}{\partial G} - \frac{\partial v^i}{\partial B^i} \cdot \frac{\partial V^{*i}}{\partial G} / \frac{\partial V^{*i}}{\partial B^i} = \frac{\partial v^i}{\partial G} - \frac{\partial v^i}{\partial B^i} \cdot MRS^i_{GB}.$$

Now, social welfare maximization problem with this indirect utility function is:

$$(3-15) \quad \Lambda = V^{*1}(B^1, Y^1, p_v, G) + \beta[V^{*2}(B^2, Y^2, p_v, G) - \bar{V}^{*2}] \\ + \rho[V^{*2}(B^2, Y^2, p_v, G) - V^{*2}(B^1, Y^1, p_v, G)] + \mu[N^2(Y^2 - B^2) + \\ N^1(Y^1 - B^1) + p_v(N^1 \cdot v^1 + N^2 \cdot v^2) - m \cdot G],$$

and the first order conditions are

$$(A-8) \quad \frac{\partial \Lambda}{\partial B^1} = \frac{\partial V^{*1}}{\partial B^1} - \rho \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} - \mu \cdot N^1 + \mu \cdot p_v \cdot N^1 \cdot \frac{\partial v^1}{\partial B^1} = 0$$

$$(A-9) \quad \frac{\partial \Lambda}{\partial Y^1} = \frac{\partial V^{*1}}{\partial Y^1} - \rho \cdot \frac{\partial \hat{V}^{*2}}{\partial Y^1} + \mu \cdot N^1 + \mu \cdot p_v \cdot N^1 \cdot \frac{\partial v^1}{\partial Y^1} = 0$$

$$(A-10) \quad \frac{\partial \Lambda}{\partial B^2} = \beta \cdot \frac{\partial V^{*2}}{\partial B^2} + \rho \cdot \frac{\partial V^{*2}}{\partial B^2} - \mu \cdot N^2 + \mu \cdot p_v \cdot N^2 \cdot \frac{\partial v^2}{\partial B^2} = 0$$

$$(A-11) \quad \frac{\partial \Lambda}{\partial Y^2} = \beta \cdot \frac{\partial V^{*2}}{\partial Y} + \rho \cdot \frac{\partial V^{*2}}{\partial Y} + \mu \cdot N^2 + \mu \cdot p_v \cdot N^2 \cdot \frac{\partial v^2}{\partial Y} = 0$$

$$(A-12) \quad \frac{\partial \Lambda}{\partial G} = \frac{\partial V^{*1}}{\partial G} + \beta \frac{\partial V^{*2}}{\partial G} + \rho \frac{\partial V^{*2}}{\partial G} - \rho \frac{\partial \hat{V}^{*2}}{\partial G} - \mu \cdot m + \mu \cdot p_v [N^1 \cdot \frac{\partial v^1}{\partial G} + N^2 \cdot \frac{\partial v^2}{\partial G}] = 0$$

Adding and subtracting  $\rho \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \cdot \frac{\frac{\partial V^{*1}}{\partial G}}{\frac{\partial V^{*1}}{\partial B^1}}$ , we get from equation (A16):

$$\begin{aligned} (A-12') \quad \frac{\partial \Lambda}{\partial G} &= \frac{\partial V^{*1}}{\partial G} + \beta \frac{\partial V^{*2}}{\partial G} + \rho \frac{\partial V^{*2}}{\partial G} - \rho \frac{\partial \hat{V}^{*2}}{\partial G} - \mu \cdot m + \mu \cdot p_v [N^1 \cdot \frac{\partial v^1}{\partial G} + N^2 \cdot \frac{\partial v^2}{\partial G}] \\ &= \left[ \frac{\partial V^{*1}}{\partial B^1} - \rho \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \right] \cdot \frac{\frac{\partial V^{*1}}{\partial G}}{\frac{\partial V^{*1}}{\partial B^1}} + (\beta + \rho) \cdot \frac{\partial V^{*2}}{\partial B^2} \cdot \frac{\frac{\partial V^{*2}}{\partial G}}{\frac{\partial V^{*2}}{\partial B^2}} + \rho \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \cdot \left[ \frac{\frac{\partial V^{*1}}{\partial G}}{\frac{\partial V^{*1}}{\partial B^1}} - \frac{\frac{\partial \hat{V}^{*2}}{\partial G}}{\frac{\partial \hat{V}^{*2}}{\partial B^1}} \right] \\ &\quad - \mu \cdot m + \mu \cdot p_v [N^1 \cdot \frac{\partial v^1}{\partial G} + N^2 \cdot \frac{\partial v^2}{\partial G}] \\ &= \left[ \frac{\partial V^{*1}}{\partial B^1} - \rho \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \right] \cdot MRS^1_{GB} + (\beta + \rho) \cdot \frac{\partial V^{*2}}{\partial B^2} \cdot MRS^2_{GB} + \\ &\quad \rho \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \cdot [MRS^1_{GB} - \hat{MRS}^2_{GB}] - \mu \cdot m + \mu \cdot p_v [N^1 \cdot \frac{\partial v^1}{\partial G} + N^2 \cdot \frac{\partial v^2}{\partial G}] \\ &= [\mu \cdot N^1 - \mu \cdot p_v \cdot N^1 \cdot \frac{\partial v^1}{\partial B^1}] \cdot MRS^1_{GB} + [\mu \cdot N^2 - \mu \cdot p_v \cdot N^2 \cdot \frac{\partial v^2}{\partial B^2}] \cdot MRS^2_{GB} + \\ &\quad \rho \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \cdot [MRS^1_{GB} - \hat{MRS}^2_{GB}] - \mu \cdot m + \mu \cdot p_v [N^1 \cdot \frac{\partial v^1}{\partial G} + N^2 \cdot \frac{\partial v^2}{\partial G}] \\ &= 0. \end{aligned}$$

The last equality follows directly from applying FOC conditions (A-8) and (A-10).

By dividing equation (A-12') by  $\mu$  and rearranging, we get:

$$\begin{aligned}
\text{(A-12'')} \quad & N^1 \cdot MRS^1_{GB} + N^2 \cdot MRS^2_{GB} + \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \cdot [MRS^1_{GB} - \hat{MRS}^2_{GB}] \\
& = m - p_v [N^1 \cdot \{ \frac{\partial v^1}{\partial G} - \frac{\partial v^1}{\partial B^1} \cdot MRS_1 \} + N^2 \cdot \{ \frac{\partial v^2}{\partial G} - \frac{\partial v^2}{\partial B^2} \cdot MRS^2_{GB} \}] \\
& = m - p_v [N^1 \cdot \frac{\partial v^{1c}}{\partial G} + N^2 \cdot \frac{\partial v^{2c}}{\partial G}]
\end{aligned}$$

The last equality follows from applying the Slutsky equation for  $v$ . Therefore,

$$\Sigma MRS^i_{GB} = m + \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^{*2}}{\partial B^1} \cdot [\hat{MRS}^2_{GB} - MRS^1_{GB}] - p_v \Sigma \frac{\partial v^{ic}}{\partial G}.$$

From the identity  $V^i(B^i, Y^i, p_g, G) = V^{*i}(B^i, Y^i, p_v, G)$ , I get  $\frac{\partial V^{*i}}{\partial B^i} = \frac{\partial V^i}{\partial B^i}$ . Thus,

$$\Sigma MRS^i_{GB} = m + \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}^2}{\partial B^1} \cdot [\hat{MRS}^2_{GB} - MRS^1_{GB}] - p_v \Sigma \frac{\partial v^{ic}}{\partial G}. \quad \mathbf{Q.E.D.}$$

### Appendix 3: proof of Proposition 5

I will follow the same trick as Gaube (2005) with slight differences. Suppose a fictitious economy with (fictitious) prices and transfer are given below

- 1) price  $q_c$  for the private consumption
- 2) price  $q_v$  for the access to the public good
- 3) price  $q_L^i$  for leisure (i = 1, 2)
- 4) price  $q_G^i$  for public good
- 5) lump-sum transfer (or tax)  $I^i$

Then, Individual problem is <sup>35</sup>

$$\max U^i (c^i, v^i, G, H^i) \quad \text{subject to } q_c \cdot c^i + q_G^i \cdot G + q_v \cdot v^i = q_L^i \cdot H^i + I^i.$$

**[The first step]  $G^Z > G^S$**

In this fictitious economy, second-best optimum can be attained as an equilibrium if the government chooses prices and transfers  $(q_c^S, q_v^S, q_L^{iS}, q_G^{iS}, I^{iS})$  such that

$$\frac{q_L^{iS}}{q_c^S} = -\frac{U_H^{iS}}{U_c^S}, \quad \frac{q_G^{iS}}{q_c^S} = \frac{U_G^{iS}}{U_c^S}, \quad \frac{q_v^S}{q_c^S} = \frac{U_v^S}{U_c^S} \quad \text{and } I^{iS} = q_c^S \cdot c^{iS} + q_G^{iS} \cdot G^S + q_v^S \cdot v^{iS} -$$

$q_L^{iS} \cdot H^{iS}$ . Also, the intermediate optimum can also be attained with  $(q_c^Z, q_v^Z, q_L^{iZ},$

$$q_G^{iZ}, I^{iZ}) \text{ such that } \frac{q_L^{iZ}}{q_c^Z} = -\frac{U_H^{iZ}}{U_c^{iZ}}, \quad \frac{q_G^{iZ}}{q_c^Z} = \frac{U_G^{iZ}}{U_c^{iZ}}, \quad \frac{q_v^Z}{q_c^Z} = \frac{U_v^{iZ}}{U_c^{iZ}} \quad \text{and } I^{iZ} = q_c^Z \cdot c^{iZ} +$$

$q_G^{iZ} \cdot G^Z + q_v^Z \cdot v^{iZ} - q_L^{iZ} \cdot H^{iZ}$ .

From Fact 1 on p. 36, in an economy without self-selection constraints, the

marginal tax rate for each type of individual is zero, so  $\frac{q_L^{iZ}}{q_c^Z} = w^i$  for  $i = 1, 2$ . However,

in a second-best economy with the self-selection constraint, a type-1 individual's

marginal tax rate is positive, while that of type-2 individual is zero,  $\frac{q_L^{1S}}{q_c^S} < w^1$  and

$$\frac{q_L^{2S}}{q_c^S} = w^2. \quad \text{Therefore, } \frac{q_L^{1S}}{q_c^S} < w^1 = \frac{q_L^{1Z}}{q_c^Z} \quad \text{and} \quad \frac{q_L^{2S}}{q_c^S} = w^2 = \frac{q_L^{2Z}}{q_c^Z}.$$

---

<sup>35</sup> The original problem was to maximize  $U^i (c^i, v^i, G, H^i)$ , but here two variables,  $v$  and  $G$ , are considered separately.

Hence, by setting  $q_c^S = q_c^Z$  without loss of generality, we get  $q_L^{1S} < q_L^{1Z}$  and  $q_L^{2S} = q_L^{2Z}$ . We also get  $U_c^{iS} = q_c^S \cdot V_B^{iS}$  and  $U_c^{iZ} = q_c^Z \cdot V_B^{iZ}$ . Also,  $U_G^{iS} = G_G^{iS}$  and  $U_G^{iZ} = G_G^{iZ}$ .<sup>36</sup> From  $m \leq \sum MRS_{GB}^i = \sum \frac{V_G^{iS}}{V_B^{iS}} = \sum q_c^S \cdot \frac{U_G^{iS}}{U_c^{iS}} = \sum q_c^S \cdot \frac{q_G^{iS}}{q_c^{iS}} = \sum q_G^{iS}$  and  $m = \sum MRS_{GB}^i = \sum q_G^{iZ}$ , we can get  $\sum q_G^{iZ} \leq \sum q_G^{iS}$ , which means either  $q_G^{1Z} \leq q_G^{1S}$  or  $q_G^{2Z} \leq q_G^{2S}$  should hold.

First assuming  $q_G^{1Z} \leq q_G^{1S}$ , consider two Hicksian demand terms below:

$h_G^1(q_c^S, q_v^S, q_L^{1S}, q_G^{1S}, U^{1S})$  and  $h_G^1(q_c^Z, 0, q_L^{1Z}, q_G^{1Z}, U^{1Z})$ . Note that  $q_c^S = q_c^Z$  and  $q_v^S > q_v^Z = 0$  and  $q_L^{1S} < q_L^{1Z}$  and  $q_G^{1S} \geq q_G^{1Z}$  and  $U^{1S} = U^{1Z}$ . Therefore,  $G^Z = h_G^1(q_c^Z, 0, q_L^{1Z}, q_G^{1Z}, U^{1Z})$ ,  $i = 1, 2$ , is greater than  $G^S = h_G^1(q_c^S, q_v^S, q_L^{1S}, q_G^{1S}, U^{1S})$ ,  $i = 1, 2$ , as long as  $G$  is a Hicksian substitute for leisure and not a Hicksian substitute for  $v$ .

Next, assuming  $q_G^{2Z} \leq q_G^{2S}$ , two Hicksian demand terms become:

$h_G^2(q_c^S, q_v^S, q_L^{2S}, q_G^{2S}, U^{2S})$  and  $h_G^2(q_c^Z, 0, q_L^{2Z}, q_G^{2Z}, U^{2Z})$ . Note that  $q_c^S = q_c^Z$  and  $q_v^S > q_v^Z = 0$  and  $q_L^{2S} = q_L^{2Z}$  and  $q_G^{2S} \geq q_G^{2Z}$  and  $U^{2S} = U^{2Z}$ . Therefore,  $G^Z = h_G^2(q_c^Z, 0, q_L^{2Z}, q_G^{2Z}, U^{2Z})$ ,  $i = 1, 2$ , is greater than  $G^S = h_G^2(q_c^S, q_v^S, q_L^{2S}, q_G^{2S}, U^{2S})$ ,  $i = 1, 2$ , as long as  $G$  is a Hicksian substitute for leisure and not a Hicksian substitute for  $v$ .

---

<sup>36</sup> These two equalities follow from the envelop theorem.

**[The Second step]  $G^F \geq G^Z$**

The first-best optimum can be attained in this fictitious economy as an equilibrium if the government chooses prices and transfers  $(q_c^F, q_v^F, q_L^{iF}, q_G^{iF}, I^{iF})$

such that  $\frac{q_L^{iF}}{q_c^F} = -\frac{U_H^{iF}}{U_c^F}$ ,  $\frac{q_G^{iF}}{q_c^F} = \frac{U_G^{iF}}{U_c^F}$ ,  $\frac{q_v^{iF}}{q_c^F} = \frac{U_v^{iF}}{U_c^F}$  and  $I^{iF} = q_c^F \cdot c^{iF} + q_G^{iF} \cdot$

$$G^F + q_v^F \cdot v^{iF} - q_L^{iF} \cdot H^{iF}.$$

Again from Fact 1,  $\frac{q_L^{1F}}{q_c^F} = \frac{q_L^{1Z}}{q_c^Z} = w^1$  and  $\frac{q_L^{2F}}{q_c^F} = \frac{q_L^{2Z}}{q_c^Z} = w^2$ . Setting  $q_c^F = q_c^Z$

without loss of generality, we get  $q_L^{1F} = q_L^{1Z}$  and  $q_L^{2F} = q_L^{2Z}$ . From  $m = \sum MRS_{GB}^{iF} = \sum q_G^{iF}$  and  $m = \sum MRS_{GB}^{iZ} = \sum q_G^{iZ}$ , we get  $\sum q_G^{iF} = \sum q_G^{iZ}$ .

First, assume  $q_G^{iF} = q_G^{iZ}$  for  $i = 1, 2$ . Since we know that  $U^{iS} = U^{iZ} \leq U^{iF}$ ,  $h_G^i(q_c^F, 0, q_L^{iF}, q_G^{iF}, U^{iF})$  is greater than  $h_G^i(q_c^Z, 0, q_L^{iZ}, q_G^{iZ}, U^{iZ})$  if the Hicksian demand for  $G$  increases with the utility level. Therefore, by equivalence between Hicksian demand and Marshallian demand using expenditure function, if the (Marshallian) demand for  $G$  increases with the income level,  $G^F \geq G^Z$ .

Second, assume  $q_G^{jF} > q_G^{jZ}$ . Then  $q_G^{iF} < q_G^{iZ}$  ( $i \neq j$ ). Then,  $h_G^i(q_c^F, 0, q_L^{iF}, q_G^{iF}, U^{iF})$  is greater than  $h_G^i(q_c^Z, 0, q_L^{iZ}, q_G^{iZ}, U^{iZ})$  if the Hicksian demand for  $G$  increases with the utility level. With the same logic as above, we get  $G^F \geq G^Z$ .

By considering both steps, I have proved Proposition 5.

**Q.E.D**

## Appendix 4: proof of Proposition 6

With the similar logic to Appendix 3, suppose a fictitious economy with prices and transfer, and derive sufficient conditions guaranteeing public good provision is decreasing for both steps: 1) the first step, second-best economy to the intermediate economy; 2) the second step, the intermediate economy to the first-best economy.<sup>37</sup>

[The first step]  $G^S > G^Z$

All analyses are the same until we get  $\frac{q_L^{1S}}{q_c^S} < w^1 = \frac{q_L^{1Z}}{q_c^Z}$  and  $\frac{q_L^{2S}}{q_c^S} = w^2 = \frac{q_L^{2Z}}{q_c^Z}$ .

Suppose  $q_L^{1S} = q_L^{1Z}$  without loss of generality, then we get  $q_c^S > q_c^Z$  and  $q_L^{2S} > q_L^{2Z}$ . It

can be easily shown that  $U_H^{iS} = w^i \cdot U_Y^{iS} = w^i \cdot V_Y^{iS}$  and  $U_H^{iZ} = w^i \cdot U_Y^{iZ} = w^i \cdot V_Y^{iZ}$ ,

because  $Y^i = q_L^i \cdot H^i$ .

From  $m \geq \sum MRS^i_{GY^S} = \sum \left( -\frac{V_G^{iS}}{V_Y^{iS}} \right) = \sum \left( -w^i \cdot \frac{U_G^{iS}}{U_H^{iS}} \right) = \sum w^i \cdot \frac{q_G^{iS}}{q_L^{iS}}$  and

$m = \sum MRS^i_{GY^Z} = \sum w^i \cdot \frac{q_G^{iZ}}{q_L^{iZ}}$ , we can get  $\sum w^i \cdot \frac{q_G^{iZ}}{q_L^{iZ}} \geq \sum w^i \cdot \frac{q_G^{iS}}{q_L^{iS}}$ ,

or  $\sum w^i \cdot \left( \frac{q_G^{iZ}}{q_L^{iZ}} - \frac{q_G^{iS}}{q_L^{iS}} \right) \geq 0$ , which means either  $\frac{q_G^{1Z}}{q_L^{1Z}} \geq \frac{q_G^{1S}}{q_L^{1S}}$  or  $\frac{q_G^{2Z}}{q_L^{2Z}} > \frac{q_G^{2S}}{q_L^{2S}}$ .

First, assume  $\frac{q_G^{1Z}}{q_L^{1Z}} \geq \frac{q_G^{1S}}{q_L^{1S}}$ , then  $q_G^{1Z} \geq q_G^{1S}$ . Hence, from two Hicksian demand

terms,  $h_G^1(q_c^S, q_v^S, q_L^{1S}, q_G^{1S}, U^{1S})$  and  $h_G^1(q_c^Z, 0, q_L^{1Z}, q_G^{1Z}, U^{1Z})$ , we get  $q_c^S >$

---

<sup>37</sup> We also follow the way of Gaube (2005).

$q_c^Z$  and  $q_v^S > q_v^Z = 0$  and  $q_L^{1S} = q_L^{1Z}$  and  $q_G^{1S} \geq q_G^{1Z}$  and  $U^{1S} = U^{1Z}$ . Therefore,  $G^Z = h_G^i(q_c^Z, 0, q_L^{iZ}, q_G^{iZ}, U^{iZ})$ ,  $i = 1, 2$ , is less than  $G^S = h_G^i(q_c^S, q_v^S, q_L^{iS}, q_G^{iS}, U^{iS})$ ,  $i = 1, 2$ , as long as  $G$  is a Hicksian substitute for private consumption and not a Hicksian complements to  $v$ .

Next, assume  $\frac{q_G^{2Z}}{q_L^{2Z}} > \frac{q_G^{2S}}{q_L^{2S}}$ . Consider then two *modified* Hicksian demand terms

using the property of homogeneity of degree zero in price,

$$h_G^2(q_c^S, q_v^S, q_L^{2S}, q_G^{2S}, U^{2S}) = h_G^2\left(\frac{q_c^S}{q_L^{2S}}, \frac{q_v^S}{q_L^{2S}}, 1, \frac{q_G^{2S}}{q_L^{2S}}, U^{2S}\right) \text{ and}$$

$$h_G^2(q_c^Z, 0, q_L^{2Z}, q_G^{2Z}, U^{2Z}) = h_G^2\left(\frac{q_c^Z}{q_L^{2Z}}, 0, 1, \frac{q_G^{2Z}}{q_L^{2Z}}, U^{2Z}\right).$$

Note that  $\frac{q_c^S}{q_L^{2S}} = \frac{q_c^Z}{q_L^{2Z}} = \frac{1}{w^2}$  and  $\frac{q_v^S}{q_L^{2S}} > 0$  and  $\frac{q_G^{2Z}}{q_L^{2Z}} \geq \frac{q_G^{2S}}{q_L^{2S}}$  and  $U^{2S} = U^{2Z}$ . Therefore,

$G^S$  is greater than  $G^Z$ , as long as  $G$  is not a Hicksian complement to  $v$ .

**[The Second step]  $G^Z \geq G^F$**

Exactly the same logic as the second step of Appendix 3 can be applied here to prove that  $G^Z \geq G^F$ , with the assumption that public good is not normal.

By considering both steps, I have proved Proposition 6.

**Q.E.D**

## Appendix 5: proof of inequality $U^{iZ} \leq U^{iF}$

First note that, in the problem characterized by (I-1) to (I-3), two constraints must be binding, thus  $U^{iS} = U^{iZ}$ . Hence all we need to show is  $U^{iS} \leq U^{iF}$ . When the government is able to impose lump-sum taxes, the first-best allocation can also be attained by maximizing an alternative problem below:

$$(F-1') \quad \text{Max}_{B^1, Y^1, B^2, Y^2, G} V^{*2}(B^2, Y^2, p_v, G) \quad \text{subject to}$$

$$(F-2') \quad V^{*1}(B^1, Y^1, p_v, G) \geq \bar{V}^{*1}$$

$$(F-3') \quad N^2 \cdot (Y^2 - B^2) + N^1 \cdot (Y^1 - B^1) + p_v \cdot (N^1 \cdot v^1 + N^2 \cdot v^2) - m \cdot G \geq 0$$

The allocation from this maximization problem ((F-1') to (F-3')) should be exactly the same as that of the original first-best economy ((F-1) to (F-3)), because the optimal price for the excludable public good should be *zero* in any non-distortionary first-best economy.<sup>38</sup>

Note that the only difference between the planner's problem in the second-best economy ((3-10') to (3-13')) and that in the first-best economy ((F-1') to (F-3')) is the existence of the self-selection constraint (i.e. the constraint (3-12')), and that all the other constraints are binding in each economy. Therefore, utility levels in the first-best economy should not be less than those in the second-best economy. **Q.E.D.**

---

<sup>38</sup> In fact, by solving the problem of ((F-1') to (F-3')), it can easily be shown that the optimal  $p_v$  is zero.

**Appendix 6: proof of  $\hat{MRS}^2_{GY} < MRS^1_{GY}$  implying  $\hat{MRS}^2_{GB} < MRS^1_{GB}$**

With  $Y$  being the numeraire, the modified Samuelson Condition is

$$\sum MRS^i_{GY}{}^S = m + \frac{\rho}{\mu} \cdot \left[ -\frac{\partial \hat{V}_2}{\partial Y} \right] \cdot [\hat{MRS}^2_{GY} - MRS^1_{GY}] - \tilde{p}_v \sum \frac{\partial v_i^c}{\partial G}.$$

Therefore, if  $\hat{MRS}^2_{GY} < MRS^1_{GY}$ , then

$$\begin{aligned} 0 &> \frac{\rho}{\mu} \cdot \left[ -\frac{\partial \hat{V}_2}{\partial Y} \right] \cdot [\hat{MRS}^2_{GY} - MRS^1_{GY}] \\ &= \sum MRS^i_{GY}{}^S - m + \tilde{p}_v \sum \frac{\partial v_i^c}{\partial G} \\ &> \sum MRS^i_{GB}{}^S - m + \tilde{p}_v \sum \frac{\partial v_i^c}{\partial G} \\ &= \frac{\rho}{\mu} \cdot \frac{\partial \hat{V}_2}{\partial B} \cdot [\hat{MRS}^2_{GB} - MRS^1_{GB}]. \end{aligned}$$

Hence, it follows that  $\hat{MRS}^2_{GB} < MRS^1_{GB}$ .

**Q.E.D.**

## References

Atkinson, A. B. and N. Stern, "Pigou, Taxation and Public Goods," *Review of Economic Studies* 41 (1974), 119-128

Blomquist, S. and V. Christiansen, "The Role of Prices for Excludable Public Goods," *International Tax and Public Finance* 12 (2005), 61-79

Boadway, R. and M. Keen, "Public Goods, Self-Selection and Optimal Income Taxation," *International Economic Review* 34 (1993), 463-478

Burns, M. E. and C. Walsh, "Market Provision of Price-excludable Public Goods: A General Analysis," *Journal of Political Economy* 89 (1981), 166-191

Christiansen, V., "Which Commodity Taxes Should Supplement the Income Tax?" *Journal of Public Economics* 24 (1984), 195-220

Coase, R., "The Lighthouse in Economics," *Journal of Law and Economics* 17 (1974), 357-76

Corlett, W. J. and D. C. Hague, "Complementarity and the Excess Burden of Taxation," *Review of Economic Studies* 21 (1953-54), 21-30

Edwards, J., M. Keen and M. Tuomala, "Income Tax, Commodity Taxes and Public Good Provision: A brief guide," *Finanzarchiv* 51 (1994), 472-487

Fraser, C., "On the Provision of Excludable Public Goods," *Journal of Public Economics* 60 (1996), 111-130

Gaube, T., "Financing Public Goods with Income Taxation: Provision Rules vs. Provision Level," *International Tax and Public Finance* 12 (2005), 319-334

Goldman, S. M. and H. Uzawa, "A Note on Separability in Demand Analysis," *Econometrica* 32 (1964), 387-398

King, M. A., "A Pigovian Rule for the Optimal Provision of Public Goods," *Journal of Public Economics* 30 (1986), 273-291

Oakland, W. H., "Public Goods, Perfect Competition and Underproduction," *Journal of Political Economy* 82 (1974), 927-939

Pigou, A. C., *A Study in Public Finance* (London: Macmillan, 1947)

Rosen, H. S., *Public Finance* (5<sup>th</sup> ed., Boston: Irwin/McGraw-Hill, 1999)

Samuelson, P. A., "The Pure Theory of Public Expenditure," *Review of Economics and Statistics* 36 (1954), 387-89

Samuelson, P. A., "Diagrammatic Exposition of a Theory of Public Expenditure," *Review of Economics and Statistics* 37 (1955), 350-56

Stiglitz, J. E., "Self-Selection and Pareto Efficient Taxation," *Journal of Public Economics* 17 (1982), 213-240